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SUBGRADIENT DUALITY IN FRACTIONAL PROGRAMMING

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A duality theory for a class of non-linear fractional programmes in terms of subgradients is developed. A fractional programme which is non convex is convexified by using generalized Charnes Cooper transformation and a dual of original problem in terms of subgradients is obtained with the help of the resulting convex programme.

1. INTRODUCTION

The following fractional programme is considered

(P) $\quad$ Minimize $\dfrac{f_1(x)}{f_2(x)}$

subject to $g_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m$

where f_1, f_2 and g_i are convex finite valued functions on some finite dimensional inner product space X. (These conditions on f_1, f_2 and g_i imply that they are continuous and possess subgradients everywhere.) It is assumed that $f_1(x) \leqslant 0$ and $f_2(x) > 0$ on S, the set of all feasible solutions of (P). The functions need not be differentiable.

In section 2 of this paper, the generalized Charness Cooper transformation $y = [f_2(x)]^{-1} \cdot x$, $t = [f_2(x)]^{-1}$, is applied to the problem (P) and the following convex equivalent programme (P′) is obtained

(P′) $\quad$ Minimize $t f_1\left(\dfrac{y}{t}\right)$

subject to $t f_2\left(\dfrac{y}{t}\right) \leqslant 1$

$t g_i\left(\dfrac{y}{t}\right) \leqslant 0, \quad i = 1, 2, \ldots, m$

$t > 0.$

In section 3, the following dual problem (D) of the problem (P) is formulated

(D) $\quad$ Maximize $(-v)$
subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i\partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

$$u \geqslant 0,\ v \geqslant 0,\ w > 0$$

and the weak duality theorem and direct duality theorem are proved with the help of convex equivalent programme (P′) of (P). At the end of section 3, the following fractional programme is considered

(P_1) Minimize $\dfrac{f_1(x) + s(x \mid C)}{f_2(x)}$

subject to $g_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m$

where $s(x \mid C)$ is the support function of the compact convex set C defined by $s(x \mid C) = \text{Sup}\,\{\langle x, y \rangle \mid y \in C\}$, $f_2(x) > 0$ and $f_1(x) + s(x \mid C) \leqslant 0$ on S.

And the following dual (D_1) of (P_1) is obtained by using the formulation of the dual problem (D) of (P):

(D_1) Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(ws\left(\frac{z}{w} \,\middle|\, C\right)\right) + \sum_{i=1}^{m} u_i\partial\left(wg_i\left(\frac{z}{w}\right)\right)$$

$$+ v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

$$u \geqslant 0,\ v \geqslant 0,\ w > 0.$$

In section 4, the dual problems studied by Schechter (1977) are obtained as special cases of the dual problems (D) and (D_1).

2. Equivalence of the Problems (P) and (P′)

In this section we shall show that the programme (P′) obtained from (P) by means of generalized Charnes Cooper transformation

$$y = [f_2(x)]^{-1} \cdot x, \quad t = [f_2(x)]^{-1} \qquad \ldots(1)$$

is a convex programme equivalent to the programme (P) in some sense. The fact that (P′) is a convex programme is proved by Schiable (1976).

Theorem 2.1 — The problem (P) has an optimal solution if and only if (P′) has one and the optimal solutions of the problems (P) and (P′) are connected by (1).

Proof : Let (P) have an optimal solution x^*. We assert that

$$y^* = [f_2(x^*)]^{-1} \cdot x^*, \; t^* = [f_2(x^*)]^{-1} \qquad \ldots(2)$$

is an optimal solution of the problem (P′).

The feasibility of (y^*, t^*) for the problem (P′) clearly follows from the feasibility of x^* for the problem (P). Let (y, t) be any other feasible solution for the problem (P′). Then $x = y/t$ is feasible for the problem (P)

and $$\frac{f_1(x^*)}{f_2(x^*)} \leqslant \frac{f_1(x)}{f_2(x)} .$$

This gives that $\dfrac{f_1(x^*)}{f_2(x^*)} \leqslant t f_1\left(\dfrac{y}{t}\right)$ because of the feasibility of (y, t) and the assumption that $f_1(x) \leqslant 0$ for all x in S. On using (2) we get,

$$t^* f_1\left(\frac{y^*}{t^*}\right) \leqslant t f_1\left(\frac{y}{t}\right)$$

which shows that (y^*, t^*) is an optimal solution of the problem (P′).

Conversely let the problem (P′) have an optimal solution (y^*, t^*). Then we shall show that $x^* = y^*/t^*$ is an optimal solution of the problem (P). Clearly x^* is feasible for the problem (P). Let x be any other feasible solution of (P), then

$$y = [f_2(x)]^{-1} \cdot x, \quad t = [f_2(x)]^{-1} \qquad \ldots(3)$$

is feasible for (P′) and

$$t^* f_1\left(\frac{y^*}{t^*}\right) \leqslant t f_1\left(\frac{y}{t}\right) = \frac{f_1(x)}{f_2(x)} \qquad \ldots(4)$$

on using (3). Also

$$\frac{f_1(x^*)}{f_2(x^*)} \leqslant t^* f_1\left(\frac{y^*}{t^*}\right) \qquad \ldots(5)$$

because of the feasibility of (y^*, t^*) for (P′) and the assumption that $f_1(x) \leqslant 0$ for all x in S.

From (4) and (5), it follows that x^* is an optimal solution of the problem (P). Thus the problems (P) and (P′) are equivalent.

Remark 2.1 : Following Schiable (1976), we make the assumption that the inequality $\text{Inf}\,\{f_1(x) \mid x \in S\} < 0$ holds.

3. Duality Theorems

In this section, it is proposed to associate dual programme of the problem (P) and establish the duality theorems in terms of subgradients. Also a dual programme of the problem (P_1) is constructed in terms of subgradients. The subgradient of a convex function f defined on X at $u \in X$ is denoted by $\partial f(u)$ and is given by

$$\partial f(u) = \{p \in X \mid f(x) - f(u) \geqslant p(x-u) \text{ for all } x \in X\}.$$

We assert that dual of (P) is the following programme :

(D) Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

$$u \geqslant 0, \quad v \geqslant 0, \quad w > 0.$$

Since f_1, f_2 and g_i are convex and $w > 0$, therefore $F_1(z, w) = wf_1\left(\frac{z}{w}\right)$, $G_i(z, w) = wg_i\left(\frac{z}{w}\right)$ for $i = 1, 2, \ldots, m$ and $F_2(z, w) = wf_2\left(\frac{z}{w}\right)$ are convex functions defined on the convex set $T = \{(z, w) \in X \times R \mid w > 0\}$ as shown by Schiable (1976) and their subgradients are defined.

Theorem 2.1 (Weak duality) — The infimum of (P) is greater than or equal to supremum of (D).

PROOF : Let x be feasible for (P) and let (z, w, u, v) be feasible for (D). Then $u \geqslant 0$, $v \geqslant 0$, $w > 0$ and

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

i.e. $$(0, 0) = (y, y_0) + \sum_{i=1}^{m} u_i(y_i, y_{i_0}) + v(k, k_0) \quad \ldots(6)$$

where $(y, y_0) \in \partial\left(wf_1\left(\frac{z}{w}\right)\right)$, $(y_i, y_{i_0}) \in \partial\left(wg_i\left(\frac{z}{w}\right)\right)$ for $i = 1, 2, \ldots, m$,

and $(k, k_0) \in \partial\left(wf_2\left(\frac{z}{w}\right)\right)$.

(6) gives that

$$0 = y + \sum_{i=1}^{m} u_i y_i + vk \text{ and } 0 = y_0 + \sum_{i=1}^{m} u_i y_{i_0} + vk_0. \quad \ldots(7)$$

It is well known that (Rockafellar 1970)

$$(y, y_0) \in \partial(F_1(z, w)) \Leftrightarrow F_1(z, w) + F_1^*(y, y_0) = \langle z, y \rangle + wy_0$$

$$\text{for all } (z, w) \in T$$

where F_1^* stands for the conjugate function of F_1. Also the conjugates of the functions $F_1(z, w)$, $G_i(z, w)$ and $F_2(z, w)$ are identically zero functions (Peterson 1976 and Scott and Jefferson 1980).

Thus we get $(y, y_0) \in \partial \left(wf_1\left(\frac{z}{w}\right)\right)$ if and only if

$$wf_1\left(\frac{z}{w}\right) = \langle z, y \rangle + wy_0 \text{ for all } (z, w) \in T. \qquad \ldots(8)$$

Similarly, $$wg_i\left(\frac{z}{w}\right) = \langle z, y_i \rangle + wy_{i_0} \text{ for all } (z, w) \in T \text{ and } i = 1, 2, \ldots, m \qquad \ldots(9)$$

and $$wf_2\left(\frac{z}{w}\right) = \langle z, k \rangle + wk_0 \quad \text{for all } (z, w) \in T. \qquad \ldots(10)$$

Let $$w^0 = [f_2(x)]^{-1} \text{ and } z^0 = [f_2(x)]^{-1}.x. \qquad \ldots(11)$$

Then $(z^0, w^0) \in T$ and (8) gives

$$w^0 f_1\left(\frac{z^0}{w^0}\right) = \langle z^0, y \rangle + w^0 y_0.$$

It follows from here and (11) that

$$\begin{aligned}
\frac{f_1(x)}{f_2(x)} &= \langle z^0, y \rangle + w^0 y_0 \\
&= \langle z^0, -\sum_{i=1}^{m} u_i y_i - vk \rangle + w^0\left(-\sum_{i=1}^{m} u_i y_{i_0} - vk_0\right) \text{ [By (7)]} \\
&= \sum_{i=1}^{m} u_i [\langle z^0, y_i \rangle + w^0 y_{i_0}] - v[\langle z^0, k \rangle + w^0 k_0] \\
&= -\sum_{i=1}^{m} u_i w^0 g_i\left(\frac{z^0}{w^0}\right) - vw^0 f_2\left(\frac{z^0}{w^0}\right) \text{ [By (9) and (10)]} \\
&= -\sum_{i=1}^{m} u_i w^0 g_i(x) - v \qquad \text{[By (11)]} \\
&\geqslant -v \text{ because } g_i(x) \leqslant 0,\ w^0 > 0,\ u \geqslant 0
\end{aligned}$$

and the weak duality result follows.

Theorem 2.2 (Direct duality) — Let x^* be an optimal solution of the problem (P) and let us suppose that $S_1 = \{x : g_i(x) < 0,\ i = 1, 2, \ldots, m\} \neq \phi$ and $f_2(\bar{x}) < f_2(x^*)$ for some $\bar{x} \in S_1$ then there exists $u^* \geqslant 0$, $v^* \geqslant 0$ such that (z^*, w^*, u^*, v^*) is an optimal solution of (D) where $z^* = [f_2(x^*)]^{-1} x^*$, $w^* = [f_2(x^*)]^{-1}$ and the two problems have the same extremal values.

PROOF : Since x^* is an optimal solution of the problem (P), therefore, by Theorem 2.1, (z^*, w^*) is an optimal solution of the problem (P') where $z^* = [f_2(x^*)]^{-1} x^*$, $w^* = [f_2(x^*)]^{-1}$. Also $S_1 \neq \phi$ and $f_2(\bar{x}) < f_2(x^*)$ for some $\bar{x} \in S_1$.
Therefore,

$$\left\{(y, t) \mid tf_2\left(\frac{y}{t}\right) < 1, tg_i\left(\frac{y}{t}\right) < 0 \text{ for } i = 1, 2, \ldots, m, t > 0\right\} \neq \phi$$

as $(\bar{y} = [f_2(x^*)]^{-1} \bar{x}, \bar{t} = [f_2(x^*)]^{-1})$ belongs to this set i.e. Slater's condition is satisfied for the problem (P') which is a convex programme of the form

$$\text{Minimize } F_1(y, t)$$

$$\text{subject to } F_2(y, t) \leqslant 1$$

$$G_i(y, t) \leqslant 0, \quad i = 1, 2, \ldots, m$$

where the functions F_1, F_2 and $G_i (i = 1, 2, \ldots, m)$ are defined on the inner product space T.

Therefore, Generalized Kuhn Tucker Theorem (Schechter 1977) can be applied to problem (P'). Thus $\exists$ u^*, v^* such that

$$u^* \geqslant 0, v^* \geqslant 0$$

$$\sum_{i=1}^{m} u_i^* w^* g_i\left(\frac{z^*}{w^*}\right) + v^*\left(w^* f_2\left(\frac{z^*}{w^*}\right) - 1\right) = 0 \qquad \ldots(12)$$

and

$$0 \in \partial\left(w^* f_1\left(\frac{z^*}{w^*}\right)\right) + \sum_{i=1}^{m} u_i^* \partial\left(w^* g_i\left(\frac{z^*}{w^*}\right)\right) + v^* \partial\left(w^* f_2\left(\frac{z^*}{w^*}\right) - 1\right). \qquad \ldots(13)$$

Clearly (z^*, w^*, u^*, v^*) is feasible for (D) as $w^* > 0$

and $$\partial\left(w^* f_2\left(\frac{z^*}{w^*}\right) - 1\right) = \partial\left(w^* f_2\left(\frac{z^*}{w^*}\right)\right).$$

Let (z, w, u, v) be any other feasible solution of (D). Since x^* is feasible for (P), therefore by Theorem 3.1,

$$\frac{f_1(x^*)}{f_2(x^*)} \geqslant -v. \qquad \ldots(14)$$

From (13), we have

$$(0, 0) = (y^*, y_0^*) + \sum_{i=1}^{m} u_i^* (y_i^*, y_{i_0}^*) + v^*(k^*, k_0^*) \qquad \ldots(15)$$

where

$$w^* f_1\left(\frac{z^*}{w^*}\right) = \langle z^*, y^* \rangle + w^* y_0^* \qquad \ldots(16)$$

$$w^* g_i\left(\frac{z^*}{w^*}\right) = \langle z^*, y_i^* \rangle + w^* y_{i_0}^* \quad \text{for } i = 1, 2, \ldots, m \qquad \ldots(17)$$

$$w^* f_2\left(\frac{z^*}{w^*}\right) = \langle z^*, k^* \rangle + w^* k_0^* . \qquad \ldots(18)$$

(15) gives

$$y^* + \sum_{i=1}^{m} u_i^* y_i^* + v^* k^* = 0 \qquad \ldots(19)$$

$$y_0^* + \sum_{i=1}^{m} u_i^* y_{i_0}^* + v^* k_0^* = 0. \qquad \ldots(20)$$

Taking inner product of (19) by z^*, multiplying (20) by w^* and adding we have

$$[\langle z^*, y^* \rangle + w^* y_0^*] + \sum_{i=1}^{m} u_i^* [\langle z^*, y_i^* \rangle + y_{i_0}^* w^*] + v^* [\langle z^*, k^* \rangle + k_0^* w^*] = 0.$$

Using (16), (17) and (18), we get

$$w^* f_1\left(\frac{z^*}{w^*}\right) + \sum_{i=1}^{m} u_i^* w^* g_i\left(\frac{z^*}{w^*}\right) + v^* \left(w^* f_2\left(\frac{z^*}{w^*}\right)\right) = 0.$$

Substituting the values from (12), we have

$$w^* f_1\left(\frac{z^*}{w^*}\right) + v^* = 0$$

i.e.

$$\frac{f_1(x^*)}{f_2(x^*)} = - v^*. \qquad \ldots(21)$$

From (14) and (21), we get

$$-v^* \geqslant -v$$

which shows that (z^*, w^*, u^*, v^*) is an optimal solution of (D). Also (21) shows that problems (P) and (D) have the same extremal values.

Let us now consider the problem

(P_1) Minimize $\dfrac{f_1(x) + s(x \mid C)}{f_2(x)}$

subject to $g_i(x) \leqslant 0, \quad i = 1, 2, \ldots, m$

where $s(x \mid C)$ is the support function of the compact convex set C, f_1, f_2 and g_i are finite valued convex functions defined on $X, f_2(x) > 0$ and $f_1(x) + s(x \mid C) \leqslant 0$ on S. Let $f(x) = f_1(x) + h(x)$ where $h(x) = s(x \mid C)$ then $f(x)$ is a finite valued convex function defined on X and $f(x) \leqslant 0$ on S. Constructing the dual problem of (P_1) as prescribed in this section, we have the following dual of (P_1)

Maximize $(-v)$

subject to

$$0 \in \partial\left(wf\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right),$$

$$u \geqslant 0, \quad v \geqslant 0, \quad w > 0.$$

Now

$$\partial\left(wf\left(\frac{z}{w}\right)\right) = \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(wh\left(\frac{z}{w}\right)\right)$$

(Rockafellar 1970 Theorem 23.8)

$$= \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(ws\left(\frac{z}{w} \,\Big|\, C\right)\right).$$

Thus the dual of (P_1) is D_1 and is given by

(D_1) Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(ws\left(\frac{z}{w} \,\Big|\, C\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right)$$

$$+ v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

$$u \geqslant 0, \quad v \geqslant 0, \quad w > 0.$$

Remark 3.1 : As remarked in section 1, we assume that the inequality

$\text{Inf}\,\{f(x)/x \in S\} < 0$ holds.

4. Particular Cases

If $f_2(x)$ is affine in the problem (P) then one needs only to associate the following equivalent problem (P″) with) (P)

(P″) Minimize $tf_1\left(\frac{y}{t}\right)$

subject to $tf_2\left(\frac{y}{t}\right) = 1$

$tg_i\left(\frac{y}{t}\right) \leqslant 0, \quad i = 1, 2, \ldots, m$

$t > 0.$

This can be shown by reworking the proof of Theorem 2.1 and the assumption $f_1(x) \leqslant 0$ on S is no longer necessary (Schiable 1976). The dual of the problem (P) with $f_2(x)$ affine can be similarly shown to be

Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right)$$

$u \geqslant 0, w > 0.$

If $f_2(x) = 1$ for all $x \in X$, then the Problem (P) of this paper reduces to the problem (P) of Schechter (1977) and the dual (D) in this case becomes

Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \sum_{i=1}^{m} u_i \partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial(w) \qquad \ldots(22)$$

$u \geqslant 0, \quad w > 0.$

(22) gives that

$$(0, 0) = (y, y_0) + \sum_{i=1}^{m} u_i(y_i, y_{i_0}) + v(0, 1)$$

where $(y, y_0) \in \partial\left(wf_1\left(\frac{z}{w}\right)\right)$, $(y_i, y_{i_0}) \in \partial\left(wg_i\left(\frac{z}{w}\right)\right)$ for $i = 1, 2, \ldots, m$ and $\partial(w) = \{(0, 1)\}$.

It follows that

$$0 = y + \sum_{i=1}^{m} u_i y_i, \qquad \ldots(23)$$

$$0 = y_0 + \sum_{i=1}^{m} u_i y_{i_0} + v, \qquad \ldots(24)$$

$$wf_1\left(\frac{z}{w}\right) = \langle z, y \rangle + wy_0 \text{ for all } (z, w) \in T, \qquad \ldots(25)$$

and

$$wg_i\left(\frac{z}{w}\right) = \langle z, y_i \rangle + wy_{i_0} \text{ for each } i = 1, 2, \ldots, m \text{ and all } (z, w) \in T. \qquad \ldots(26)$$

Taking inner product of (23) by z, multiplying (24) by w, adding and using (25) and (26), we have

$$0 = wf_1\left(\frac{z}{w}\right) + \sum_{i=1}^{m} u_i wg_i\left(\frac{z}{w}\right) + vw$$

i.e.
$$-v = f_1\left(\frac{z}{w}\right) + \sum_{i=1}^{m} u_i g_i\left(\frac{z}{w}\right).$$

Taking $\frac{z}{w} = x$, we obtain

$$-v = f_1(x) + \sum_{i=1}^{m} u_i g_i(x) \text{ for all } x \in X$$

$$f_1(x) = \langle x, y \rangle + y_0 \quad \text{for all } x \in X \qquad \text{[By (25)]}$$

$$g_i(x) = \langle x, y_i \rangle + y_{i_0} \text{ for each } i = 1, 2, \ldots, m \text{ and all } x \in X \quad \text{[By (26)]}$$

These give that

$$f_1(x) - f_1(x_0) = \langle x - x_0, y \rangle \text{ for all } x_0 \in X$$

$$g_i(x) - g_i(x_0) = \langle x - x_0, y_i \rangle \text{ for each } i = 1, 2, \ldots, m \text{ and all } x_0 \in X$$

i.e.
$$y \in \partial f_1(x) \text{ and } y_i \in \partial g_i(x) \text{ for } i = 1, 2, \ldots, m.$$

Thus (23) implies that

$$0 \in \partial f_1(x) + \sum_{i=1}^{m} u_i \partial g_i(x)$$

and dual (D) can be written as

$$\text{Maximize } f_1(x) + \sum_{i=1}^{m} u_i g_i(x)$$

$$\text{subject to } 0 \in \partial f_1(x) + \sum_{i=1}^{m} u_i \partial g_i(x)$$

$$u \geqslant 0$$

which is the dual (D) of Schechter (1977).

If $f_2(x)$ is affine in the problem (P_1) then the assumption that $f_1(x) + s(x \mid C) \leqslant 0$ on S can be dropped and the dual of (P_1) can be written as

Maximize $(-v)$

subject to

$$0 \in \partial\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(ws\left(\frac{z}{w} \,\Big|\, C\right)\right) + \sum_{i=1}^{m} u_i\partial\left(wg_i\left(\frac{z}{w}\right)\right) + v\partial\left(wf_2\left(\frac{z}{w}\right)\right),$$

$$u \geqslant 0, \quad w > 0.$$

If $f_2(x) = 1$ for all $x \in X$ and $f_1(x)$ and $g_i(x)$ for $i = 1, 2, \ldots, m$, are differentiable on X, then the problem (P_1) becomes Problem (P') of Schechter (1977) and the dual problem becomes

Maximize $(-v)$

subject to

$$0 \in \nabla\left(wf_1\left(\frac{z}{w}\right)\right) + \partial\left(ws\left(\frac{z}{w} \,\Big|\, C\right)\right) + \sum_{i=1}^{m} u_i\nabla\left(wg_i\left(\frac{z}{w}\right)\right) + v\nabla(w) \qquad \ldots(27)$$

$$u \geqslant 0, \quad w > 0,$$

where ∇ stands for the gradient.

(27) can be written as

$$0 = \nabla\left(wf_1\left(\frac{z}{w}\right)\right) + (r, r_0) + \sum_{i=1}^{m} u_i\nabla\left(wg_i\left(\frac{z}{w}\right)\right) + v\nabla(w) \qquad \ldots(28)$$

where $\quad (r, r_0) \in \partial\left(ws\left(\frac{z}{w} \,\Big|\, C\right)\right) = \partial\left(wh\left(\frac{z}{w}\right)\right)$

i.e. $\quad ws\left(\frac{z}{w} \,\Big|\, C\right) = \langle z, r \rangle + wr_0$ for all $(z, w) \in T$. $\qquad \ldots(29)$

On simplifying (28), we get

$$0 = \nabla f_1\left(\frac{z}{w}\right) + \sum_{i=1}^{m} u_i \nabla g_i\left(\frac{z}{w}\right) + r \qquad \ldots(30)$$

$$0 = f_1\left(\frac{z}{w}\right) + \sum_{i=1}^{m} u_i g_i\left(\frac{z}{w}\right) + v + \left\langle \frac{z}{w}, r \right\rangle + r_0. \qquad \ldots(31)$$

Taking $\frac{z}{w} = x$ and using (29), we get from (31),

$$-v = f_1(x) + \sum_{i=1}^{m} u_i g_i(x) + s(x \mid C). \qquad \ldots(32)$$

Also (29) gives

$$s(x \mid C) = \langle x, r \rangle + r_0 \text{ for all } x \in X.$$

Therefore $s(x \mid C) - s(x_0 \mid C) = \langle x - x_0, r \rangle$ for all $x_0 \in X$ and it follows that $r \in \partial(s(x \mid C))$.

From Theorem 2 of Schechter (1977), we have

$$\partial(s(x \mid C)) = C \cap \{z \mid \langle z, x \rangle = s(x \mid C))\}$$

Thus $\quad r \in C$ and $\langle r, x \rangle = s(x \mid C)$.

On using (30) and (32), the dual (D_1) can be written as

$$\text{Maximize} \quad f_1(x) + \sum_{i=1}^{m} u_i g_i(x) + \langle r, x \rangle$$

$$\text{subject to } 0 = \nabla f_1(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) + r$$

$$r \in C, u \geqslant 0 \text{ and } \langle r, x \rangle = s(x \mid C)$$

which is the dual (D′) of Problem (P′) of Schechter (1977).

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REFERENCES

Peterson, E. L. (1976). Geometric programming. *SIAM Rev.*, **18**, 1–52.

Rockafellar, R. T. (1970). Convex Analysis, Princeton University Press, Princeton, N. J.

Schechter, M. (1977). A subgradient duality theorem. *J. Math. Anal. Appl.*, **61**, 850–55.

Schiable, S. (1976). Fractional programming I, Duality. *Manag. Sci.*, **22**, 858–62.

Scott, C. H., and Jefferson, T. R. (1980). Fractional programming duality via geometric programming duality. *J. Aust. Math. Soc.*, **21** (series B), 398–401.