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ON THE GENERALIZATION OF PACHPATTE'S NON-UNIQUE FIXED POINT THEOREM

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The result on non-unique fixed point theorem involving four points of the space has been established. This result includes as a special case the fixed point theorem recently established by Pachpatte (1979).

1. INTRODUCTION

Ćirić (1974) has first studied the non-unique fixed point theorems and then further studies have been made by Achari (1976) and Pachpatte (1979). But in all the cases the mapping under consideration involves only two points of the space under consideration. Until recently, Pittnauer (1980) has studied fixed point theorems for mapping involving four points of the space and recently Achari (1979, 1980) has extended this idea further to establish some fixed point theorems.

The aim of this paper it to establish a further generalization of the non-unique fixed point theorem recently established by Pachpatte (1979, Theorem 1) for mappings involving four points of the space.

Let (X, d) be a f-orbitally complete metric space. Let $\psi_i : \bar{P} \underset{i=1,2}{\rightarrow} [0, \infty)$ (P is the range of d and $\bar{P}$ is the closure of P) be upper semicontinuous function from the right on $\bar{P}$ and satisfies the condition

$$\psi_i(t) < t \text{ for } t > 0 \text{ and } \psi_i(0) = 0,\ i = 1, 2. \qquad ...(1)$$

Also, let f be an orbitally continuous mapping of X into itself such that

$$\min \{d(fu_1, fu_2)\, d(fu_3, fu_4),\ d(u_1, u_2)\, d(fu_3, fu_4),\ [d(u_2, fu_4)]^2\} - \min \{d(u_1, fu_3)\, d(u_2, fu_4),\ d(u_1, fu_4)\, d(u_2, fu_3)\}$$
$$\leqslant \psi_1(d(u_1, fu_3))\, \psi_2(d(u_2, fu_4)) \qquad ...(2)$$

for $u_1, u_2, u_3, u_4 \in X$.

2. MAIN RESULT

Theorem 1 — Let f be an orbitally continuous mapping of X into itself satisfying (2), then f has a fixed point.

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PROOF : Let $x, y \in X$ and define $u_1 = fy$, $u_2 = fx$, $u_3 = x$, $u_4 = y$.
Then the expression (2) takes the form

$$\min \{d(f^2x, f^2y)\, d(fx, fy),\, d(fx, fy)\, d(fx, fy),\, [d(fx, fy)]^2\}$$
$$-\min \{d(fy, fx)\, d(fx, fy),\, d(fy, fy)\, d(fx, fx)\}$$
$$\leqslant \psi_1[d(fy, fx)]\, \psi_2\, [d(fy, fx)]$$

or
$$\min \{d(f^2x, f^2y)\, d(fx, fy),\, d(fx, fy)\, d(fx, fy),\, [d(fx, fy)]^2\}$$
$$\leqslant \psi_1\, [d(fx, fy)]\, \psi_2\, [d(fx, fy)]. \qquad \ldots(3)$$

Let $x_0 \in X$ be arbitrary and we construct a sequence $\{x_n\}$ defined by

$$fx_{n-2} = x_{n-1},\, fx_{n-1} = x_n,\, fx_n = x_{n+1},\, n = 1, 2, \ldots .$$

Let us put $x = x_{n-1}$, $y = x_{n-2}$ in (3), then we have

$$\min \{d(f^2x_{n-1}, f^2x_{n-2})\, d(fx_{n-1}, fx_{n-2}),\, d(fx_{n-1}, fx_{n-2})$$
$$\times\, d(fx_{n-1}, fx_{n-2}),\, [d(fx_{n-1}, fx_{n-2})]^2\}$$
$$\leqslant \psi_1\, [d(fx_{n-1}, fx_{n-2})]\, \psi_2\, [d(fx_{n-1}, fx_{n-2})]$$

or
$$\min \{d(x_n, x_{n+1})\, d(x_n, x_{n-1}),\, d(x_n, x_{n-1})\, d(x_n, x_{n-1}),\, [d(x_n, x_{n-1})]^2\}$$
$$\leqslant \psi_1\, [d(x_n, x_{n-1})]\, \psi_2\, [d(x_n, x_{n-1})]$$

Since
$$d(x_{n-1}, x_{n-1})\, d(x_n, x_{n-1}) \leqslant \psi_1\, [d(x_n, x_{n-1})]\, \psi_2\, [d(x_n, x_{n-1})]$$
$$< d(x_n, x_{n-1})\, d(x_n, x_{n-1})$$

and
$$[d(x_n, x_{n-1})]^2 = d(x_n, x_{n-1})\, d(x_n, x_{n-1})$$
$$\leqslant \psi_1\, [d(x_n, x_{n-1})]\, \psi_2\, [d(x_n, x_{n-1})]$$

are impossible, we have

$$d(x_n, x_{n+1})\, d(x_n, x_{n-1}) \leqslant \psi_1\, [d(x_n, x_{n-1})]\, \psi_2\, [d(x_n, x_{n-1})]$$

i.e.,
$$d(x_n, x_{n+1}) \leqslant \psi_1\, [d(x_n, x_{n-1})]. \qquad \ldots(4)$$

Let us take $C_{n+1} = d(x_n, x_{n+1})$, then

$$C_{n+1} = d(x_n, x_{n+1}) \leqslant \psi_1\, [d(x_n, x_{n-1})] \leqslant \psi_1(C_n), \qquad \ldots(5)$$

From (5) it is clear that C_n decreases with n and hence $C_n \to C$ say, as $n \to \infty$.

If possible, let $C > 0$. Then since ψ_i is upper semicontinuous, we obtain in the limit as $n \to \infty$

$$C \leqslant \psi_1(C) < C$$

which is impossible unless $C = 0$.

Next, we shall show that the sequence $\{x_n\}$ is Cauchy. Suppose that it is not so. Then there exists an $\epsilon > 0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$ with $m(k) > n(k) \geqslant k$ such that

$$d_k = d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon, \; k = 1, 2, \ldots . \qquad \ldots(6)$$

If $m(k)$ is the smallest integer exceeding $n(k)$ for which (6) holds, then from well-ordering principle, we have

$$d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \qquad \ldots(7)$$

Then

$$\begin{aligned} d_k &\leqslant d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &\leqslant C_{m(k)} + \epsilon < C_k + \epsilon \end{aligned}$$

which implies $d_k \to \epsilon$ as $k \to \infty$.

Also we have

$$\begin{aligned} d_k = d(x_m, x_n) &\leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_n, x_{n+1}) \\ &\leqslant C_{m+1} + C_{n+1} + d(x_{m+1}, x_{n+1}). \end{aligned}$$

Putting $u_1 = x_n$, $u_2 = x_m$, $u_3 = x_{m-1}$, $u_4 = x_{n-1}$ in (2), we get

$$\begin{aligned} &\min \{d(fx_n, fx_m)\, d(fx_{n-1}, fx_{m-1}),\, d(x_n, x_m)\, d(fx_{n-1}, fx_{m-1}), \\ &\qquad [d(x_m, fx_{n-1})]^2\} - \min \{d(x_n, fx_{m-1})\, d(x_m, fx_{n-1}), \\ &\qquad d(x_n, fx_{n-1})\, d(x_m, fx_{m-1})\} \leqslant \psi_1 [d(x_n, fx_{m-1})]\, \psi_2 [d(x_m, fx_{n-1})] \end{aligned}$$

i.e.,
$$d(x_{n+1}, x_{m+1}) \leqslant \psi_1 [d(x_n, x_m)].$$

So
$$\begin{aligned} d_k &\leqslant C_{m+1} + C_{n+1} + \psi_1 [d(x_n, x_m)] \\ &\leqslant C_{m+1} + C_{n+1} + \psi_1(d_k). \end{aligned}$$

Letting $k \to \infty$ we get $\epsilon \leqslant \psi_1(\epsilon) < \epsilon$. Which is a contradiction if $\epsilon > 0$. This leads us to conclude that the sequence $\{x_n\}$ is Cauchy.

X being f-orbitally complete there is some $z \in X$ such that $z = \lim_{n \to \infty} f^n x$. By the orbital continuity of f we have

$$fz = \lim_n ff^n x = z.$$

Thus z is a fixed point of f.

If z and ω are fixed points of f then by putting $u_1 = u_4 = z$ and $u_2 = u_3 = \omega$ in (2) we obtain that $d(z, \omega) < d(z, \omega)$ i.e., the fixed points of f are unique.

We simply remark that under the conditions $\psi_1(t) = \alpha t$, $\psi_2(t) = t$, $0 < \alpha < 1$

and $u_1 = u_3 = x$, $x_2 = u_4 = y$ the fixed point theorem established in this paper contains as a special case the recent result of Pachpatte (1979, Theorem 1).

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