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SOME RESULTS ON FIXED POINTS IN COMPLETE METRIC SPACES

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Some results on fixed points in complete metric spaces have been obtained in the present paper.

INTRODUCTION

The existence of a unique fixed point of a contraction map in a complete metric space was established by Banach. Kannan (1968, 1969), Reich (1971), Hardy and Roggers (1973), Jaggi (1977) and many others generalized this principle in different ways. The object of the present paper is to obtain another generalization.

MAIN RESULTS

Theorem 1 — Let T be a self-map of a complete metric space (X, ρ) such that for some positive integer m and for some $\alpha_i, \beta_i > 0$ with $\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 < 1$

$$\rho(T^m x, T^m y) \leqslant \frac{\alpha_1 \rho(x, T^m x)\, \rho(y, T^m y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T^m x)\, \rho(y, T^m x)}{\rho(T^m x, T^m y)}$$

$$+ \frac{\alpha_3 \rho(x, T^m y)\, \rho(y, T^m y)}{\rho(T^m x, T^m y)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, T^m x)$$

$$+ \beta_3 \rho(y, T^m y) + \beta_4 \rho(x, T^m y) + \beta_5 \rho(y, T^m x) \qquad \ldots(A)$$

for all $x, y \in X$ with $x \neq y$, $T^m x \neq T^m y$.

If for some positive integer p, T^{mp} is continuous, then T has a unique fixed point.

PROOF : With usual arguments we can assume that $\alpha_2 = \alpha_3$, $\beta_2 = \beta_3$, $\beta_4 = \beta_5$. Let $x_0 \in X$. Let $x_n = T^{mn} x_0$ for every positive integer n. $x_n = x_{n+1}$ for some n implies that x_n is a fixed point of T^m. So we assume that $x_n \neq x_{n+1}$ for all n.

Now $\quad \rho(x_{n+1}, x_n) = \rho(T^m x_n, T^m x_{n-1})$

$$\leqslant \frac{\alpha_1 \rho(x_n, x_{n+1})\, \rho(x_{n-1}, x_n)}{\rho(x_n, x_{n-1})} + \frac{\alpha_2 \rho(x_n, x_{n+1})\, \rho(x_{n-1}, x_{n+1})}{\rho(x_{n+1}, x_n)}$$

(*equation continued on p.* 304)

$$+ \frac{\alpha_3\rho(x_n, x_n)\,\rho(x_{n-1}, x_n)}{\rho(x_{n+1}, x_n)} + \beta_1\rho(x_n, x_{n-1}) + \beta_2\rho(x_n, x_{n+1})$$

$$+ \beta_3\rho(x_{n-1}, x_n) + \beta_4\rho(x_n, x_n) + \beta_5\rho(x_{n-1}, x_{n+1}).$$

$$\therefore \qquad \rho(x_{n+1}, x_n) \leqslant k\rho(x_n, x_{n-1})$$

where $k = \dfrac{\alpha_2 + \beta_1 + \beta_3 + \beta_5}{1 - \alpha_1 - \alpha_2 - \beta_2 - \beta_5}$ and $0 < k < 1$ since $\alpha_i, \beta_i > 0$

and $\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1.$

Since X is complete, by a standard argument, there exists a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Let $n_k = kp$. Then $\lim_{k\to\infty} x_{n_k} = u$.

Since T^{mp} is continuous, $T^{mp}u = T^{mp}\ (\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} x_{n_{k+1}} = u.$

Let n be the smallest positive integer such the $T^{mn}u = u$ but $T^{mq}u \neq u$ for $q = 1,\dots, n-1$. We shall now show that $n = 1$. If possible, let $n > 1$.

Then
$$\rho(T^m u, u) = \rho(T^m(T^{mn}u), T^m(T^{m(n-1)}u))$$
$$\leqslant \frac{\alpha_1\rho(T^{mn}u, T^{m(n+1)}u)\,\rho(T^{m(n-1)}u, T^{mn}u)}{\rho(T^{mn}u, T^{m(n-1)}u)}$$
$$+ \frac{\alpha_2\rho(T^{mn}u, T^{m(n+1)}u)\,\rho(T^{m(n-1)}u, T^{m(n+1)}u)}{\rho(T^{m(n+1)}u, T^{mn}u)}$$
$$+ \frac{\alpha_3\rho(T^{mn}u, T^{mn}u)\,\rho(T^{m(n-1)}u, T^{mn}u)}{\rho(T^{m(n+1)}u, T^{mn}u)}$$
$$+ \beta_1\rho(T^{mn}u, T^{m(n-1)}u) + \beta_2\rho(T^{mn}u, T^{m(n+1)}u)$$
$$+ \beta_3\rho(T^{m(n-1)}u, T^{mn}u) + \beta_4\rho(T^{mn}u, T^{mn}u)$$
$$+ \beta_5\rho(T^{m(n-1)}u, T^{m(n+1)}u)$$

which gives

$$\rho(T^m u, u) = \rho(T^{m(n+1)}u, T^{mn}u)$$
$$\leqslant k\rho(T^{mn}u, T^{m(n-1)}u)$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$\leqslant k^n\rho(T^m u, u)$$
$$< \rho(T^m u, u) \quad (\because k < 1)$$

which is a contradiction.

If possible, let there exist another point $v \in X$ such that $T^m v = v$. Then

$$\rho(u, v) = \rho(T^m u, T^m v)$$
$$\leqslant \frac{\alpha_1 \rho(u, u)\, \rho(v, v)}{\rho(u, v)} + \frac{\alpha_2 \rho(u, u)\, \rho(v, u)}{\rho(u, v)} + \frac{\alpha_3 \rho(u, v)\, \rho(v, v)}{\rho(u, v)}$$
$$+ \beta_1 \rho(u, v) + \beta_2 \rho(u, u) + \beta_3 \rho(v, v) + \beta_4 \rho(u, v) + \beta_5 \rho(v, u)$$
$$< \rho(u, v) \quad (\because \beta_1 + \beta_4 + \beta_5 < 1)$$

which is a contradiction.

$\therefore$ u is the unique fixed point of T^m.

Then $Tu = T(T^m u) = T^m(Tu)$ shows that Tu is a fixed point of T^m. Then the unicity of the fixed point of T^m yields $Tu = u$. The unicity of the fixed point of T follows from that of T^m and the fact that any fixed point of T is a fixed point of T^m.

Note : Theorem 1 is a generalization of Theorems 1 – 3 of Jaggi (1977).

Theorem 2 — Let T_1 and T_2 be two self-maps of a complete metric space (X, ρ) satisfying the following conditions :

(i) $\rho(T_1 x, T_2 y)$

$$\leqslant \frac{\alpha_1 \rho(x, T_1 x)\, \rho(y, T_2 y)}{\rho(x, y)} + \frac{\alpha_2 \rho(x, T_1 x)\, \rho(y, T_1 x)}{\rho(T_1 x, T_2 y)}$$
$$+ \frac{\alpha_3 \rho(x, T_2 y)\, \rho(y, T_2 y)}{\rho(T_1 x, T_2 y)} + \beta_1 \rho(x, y) + \beta_2 \rho(x, T_1 x)$$
$$+ \beta_3 \rho(y, T_2 y) + \beta_4 \rho(x, T_2 y) + \beta_5 \rho(y, T_1 x)$$

for all $x, y \in X$ with $x \neq y$, $T_1 x \neq T_2 y$ and for some α_i, $\beta_i > 0$ with $\alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$, $\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1$;

(ii) $T_1 T_2$ is continuous;

(iii) there exists a $x_0 \in X$ such that $x_n \neq x_{n+1}$ for all n where

$$x_n = \begin{cases} T_1 x_{n-1} \text{ if } n \text{ be even} \\ T_2 x_{n-1} \text{ if } n \text{ be odd.} \end{cases}$$

Then T_1 and T_2 have unique fixed points and they are equal.

PROOF : We have

$$\rho(x_{2n}, x_{2n+1}) = \rho(T_1 x_{2n-1}, T_2 x_{2n})$$
$$\leqslant \frac{\alpha_1 \rho(x_{2n-1}, x_{2n})\, \rho(x_{2n}, x_{2n+1})}{\rho(x_{2n-1}, x_{2n})} + \frac{\alpha_2 \rho(x_{2n-1}, x_{2n})\, \rho(x_{2n}, x_{2n})}{\rho(x_{2n}, x_{2n+1})}$$

(*equation continued on p.* 306)

$$+ \frac{\alpha_3\rho(x_{2n-1}, x_{2n+1})\,\rho(x_{2n}, x_{2n+1})}{\rho(x_{2n}, x_{2n+1})} + \beta_1\rho(x_{2n-1}, x_{2n})$$

$$+ \beta_2\rho(x_{2n-1}\, x_{2n}) + \beta_3\rho(x_{2n}, x_{2n+1}) + \beta_4\rho(x_{2n-1}, x_{2n+1})$$

$$+ \beta_5\rho(x_{2n}, x_{2n})$$

$$\therefore \quad \rho(x_{2n}, x_{2n+1}) \leqslant k_1\rho(x_{2n-1}, x_{2n})$$

where $k_1 = \dfrac{\alpha_3 + \beta_1 + \beta_2 + \beta_4}{1 - \alpha_1 - \alpha_3 - \beta_3 - \beta_4}$ and $0 < k_1 < 1$ since

$\alpha_i, \beta_i > 0$ and $\alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1.$

Again $\rho(x_{2n+1}, x_{2n+2})$

$$= \rho(T_1\, x_{2n+1}, T_2x_{2n})$$

$$\leqslant \frac{\alpha_1\rho(x_{2n+1}, x_{2n+2})\,\rho(x_{2n}, x_{2n+1})}{\rho(x_{2n+1}, x_{2n})} + \frac{\alpha_2\rho(x_{2n+1}, x_{2n+2})\,\rho(x_{2n1}\, x_{2n+2})}{\rho(x_{2n+2}, x_{2n+1})}$$

$$+ \frac{\alpha_3\rho(x_{2n+1}, x_{2n+1})\,\rho(x_{2n}, x_{2n+1})}{\rho(x_{2n+2}, x_{2n+1})} + \beta_1\rho(x_{2n+1}. x_{2n})$$

$$+ \beta_2\rho(x_{2n+1}, x_{2n+2}) + \beta_3\rho(x_{2n}, x_{2n+1}) + \beta_4\rho(x_{2n+1}, x_{2n+1})$$

$$+ \beta_5\rho(x_{2n}, x_{2n+2})$$

$$\therefore \quad \rho(x_{2n+1}, x_{2n+2})$$

$$\leqslant \quad k_2\rho(x_{2n}, x_{2n+1})$$

where $k_2 = \dfrac{\alpha_2 + \beta_1 + \beta_3 + \beta_5}{1 - \alpha_1 - \alpha_2 - \beta_2 - \beta_5}$ and $0 < k_2 < 1$

since $\alpha_i, \beta_i > 0$ and $\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \beta_3 + 2\beta_5 < 1.$

Thus $\rho(x_n, x_{n+1}) \leqslant k_1\rho(x_{n-1}, x_n)$ or $k_2\rho(x_{n-1}, x_n)$

according as n is even or odd.

Let $k = \min. (k_1, k_2)$. Then $0 < k < 1$ and

$$\rho(x_n, x_{n+1}) \leqslant k\rho(x_{n-1}, x_n) \text{ for all } n.$$

Since X is complete, with the help of the usual argument we can find a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Let $n_k = 2k$. Then $\lim_{k\to\infty} x_{n_k} = u$.

Since T_1T_2 is continuous, $T_1T_2u = \lim_{k\to\infty} (T_1T_2\, x_{n_k}) = \lim_{k\to\infty} x_{n_{k+1}} = u.$

Then $T_2u = u$. For, if $T_2u \neq u$. then

$$\rho(T_2u, u) = \rho(T_1(T_2u), T_2u)$$

$$\leqslant \frac{\alpha_1\rho(T_2u, u)\rho(u, T_2u)}{\rho(T_2u, u)} + \frac{\alpha_2\rho(T_2u,u)\,\rho(u,u)}{\rho(u, T_2u)} + \frac{\alpha_3\rho(T_2u, T_2u)\,\rho(u,T_2u)}{\rho(u,T_2u)}$$

$$+ \beta_1\rho(T_2u, u) + \beta_2\rho(T_2u, u) + \beta_3\rho(u, T_2u) + \beta_4\rho(T_2u, T_2u)$$

$$+ \beta_5\rho(u, u)$$

$$= (\alpha_1 + \beta_1 + \beta_2 + \beta_3)\,\rho(T_2u, u)$$

$$< \rho(T_2u, u) \quad (\because \ \alpha_1 + \beta_1 + \beta_2 + \beta_3 < 1)$$

which is a contradiction.

$\therefore$ $T_2u = u$. Then $\rho(T_1u, u) = \rho(T_1(T_2u), u) = \rho(u, u) = 0$ gives $T_1u = u$.

If possible, let there exist another fixed point v of T_1.

Then $\quad \rho(v, u) = \rho(T_1v, T_2u)$

$$\leqslant \frac{\alpha_1\rho(v, v)\,\rho(u, u)}{\rho(v, u)} + \frac{\alpha_2\rho(v, v)\,\rho(u, v)}{\rho(v, u)} + \frac{\alpha_3\rho(v, u)\,\rho(u, u)}{\rho(v, u)}$$

$$+ \beta_1\rho(v, u) + \beta_2\rho(v, v) + \beta_3\rho(u, u) + \beta_4\rho(v, u) + \beta_5\rho(u, v)$$

$$< \rho(v, u) \quad (\because \ \beta_1 + \beta_4 + \beta_5 < 1)$$ which is a contradiction.

$\therefore$ u is the unique fixed point of T_1. Similarly it can be shown that u is the unique fixed point of T_2.

Note : Theorem 2 is a generalization of Theorem 4 of Jaggi (1977).

Corollary 1 — If T_1 and T_2 be replaced by T_1T_2 and T_2T_1 respectively in (i), (ii), (iii) of Theorem 2, then there exists a unique common fixed point of T_1 and T_2.

PROOF : By Theorem 2, T_1T_2 and T_2T_1 have unique fixed points Z and Z' respectively and $Z = Z'$. Then $T_1T_2\,Z = Z$ gives $T_2T_1(T_2Z) = T_2Z$ which together with the unicity of the fixed point of T_2T_1 yields $T_2Z = Z$. Similarly $T_1Z = Z$. The inequality (i) (with T_1, T_2 replaced by T_1T_2, T_2T_1) and $\beta_1 + \beta_4 + \beta_5 < 1$ yield the unicity of the common fixed point of T_1 and T_2.

Corollary 2 — Let T, T_1, T_2 be three self-maps of a complete metric space (X, ρ) satisfying the conditions (ii), (iii) of Theorem 2 together with the following conditions:

(a) $\rho(Tx, Ty) \leqslant \rho(x, y)$ for every $x, y \in X$;

(b) $TT_i = T_iT$ for $i = 1, 2$;

(c) $\rho(T_1x, T_2y) \leqslant \dfrac{\alpha_1\rho(Tx, TT_1x)\,\rho(Ty, TT_2y)}{\rho(x,y)}$

(*equation continued on p.* 308)

$$+\frac{\alpha_2\rho(Tx,TT_1x)\,\rho(Ty,TT_1x)}{\rho(T_1x,T_2y)}+\frac{\alpha_3\rho(Tx,TT_2y)\,\rho(Ty,TT_2y)}{\rho(T_1x,T_2y)}$$

$$+\beta_1\rho(x,y)+\beta_2\rho(Tx,TT_1x)+\beta_3\,\rho(Ty,TT_2y)$$

$$+\beta_4\rho(x,T_2y)+\beta_5\rho(y,T_1x)$$

for all x, $y \in X$ with $x \neq y$, $T_1x \neq T_2y$ and for some $\alpha_i, \beta_i > 0$ with $\alpha_1+2\alpha_3+\beta_1+\beta_2+\beta_3+2\beta_4<1$, $\alpha_1+2\alpha_2+\beta_1+\beta_2+\beta_3+2\beta_5<1$.

Then there exists a unique common fixed point of T, T_1 and T_2.

PROOF : (a) and (c) yield the inequality (i) of Theorem 2. $\therefore$ By Theorm 2, there exists a point $z \in X$ such that $T_iz = z$ for $i = 1,2$. Then $Tz = z$. For, if $Tz \neq z$, then

$$\rho(z,Tz)=\rho(T_1z,T_2Tz)\quad[\text{by }(b)]$$

$$\leqslant\frac{\alpha_1\rho(z,T_1z)\,\rho(Tz,T_2Tz)}{\rho(x,Tz)}+\frac{\alpha_2\rho(z,T_1z)\,\rho(Tz,T_1z)}{\rho(T_1z,T_2Tz)}$$

$$+\frac{\alpha_3\rho(z,T_2Tz)\,\rho(Tz,T_2Tz)}{\rho(T_1z,T_2Tz)}+\beta_1\rho(z,Tz)+\beta_2\rho(z,T_1z)$$

$$+\beta_3\rho(Tz,T_2Tz)+\beta_4\rho(z,T_2Tz)+\beta_5\rho(Tz,T_1z)\ [\text{by (i)}]$$

$$=(\beta_1+\beta_4+\beta_5)\,\rho(z,Tz)\ [\text{by }(b)]<\rho(z,Tz)\quad(\because \beta_1+\beta_4+\beta_5<1)$$

which is a contradiction

$\therefore$ $z = Tz$. $\therefore$ z is the unique common fiixed point of T, T_1 and T_2

Note : Taking T as the identity transformation of X one gets Theorem 2.

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