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# THE SMALLEST CUBIC MULTIGRAPHS WITH PRESCRIBED BIPARTITE, BLOCK, HAMILTONIAN AND PLANAR PROPERTIES

GEOFFREY EXOO

*Bell Laboratories, Holmdel, N. J.* 07733, *USA*

AND

FRANK HARARY*

*Churchill College, Cambridge CB*3 *ODS, England*

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A cubic multigraph may or may not be 2-connected (a block), planar, bipartite or hamiltonian. As six of these sixteen possibilites are impossible, we determine the smallest multigraphs for each of the other ten cases.

## 1. INTRODUCTION

With Asano and Saito we (Asano *et al.* 1981) constructed the smallest cubic graph (Fig. 1) which is 2-connected, planar, bipartite and not hamiltonian. Calling one graph smaller than another if it has fewer points, we showed that this graph with 26 points is the unique such smallest graph.

In the first sequel to Asano (1981), we listed in Exoo and Harary (1981) the following four 'atomic properties' $A_i$ of graphs:

$A_1$ · 2-connected

$A_2$ · planar

$A_3$ · bipartite

$A_4$ · hamiltonian.

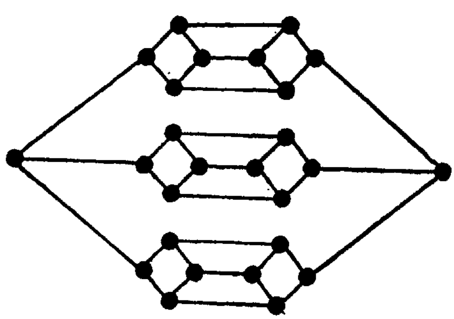

FIG. 1. The smallest cubic planar bipartite nonhamiltonian 2-connected graph.

*On sabbatical leave 1980–81 from the University of Michigan, Ann Arbor, MI 48109, USA.

Then we defined 16 'molecular properties' $i_1i_2i_3i_4$ of a graph $G$, where for $k = 1$ to 4, $i_k = 1$ if $G$ satisfies $A_k$ and $i_k = 0$ otherwise. For example, a graph with property 1111 is 2-connected, planar, bipartite and hamiltonian; the smallest such graph is the quadrilateral $C_4$, following the notation and terminology of Harary (1969). Since a cubic graph or multigraph which is hamiltonian or bipartite is necessarily 2-connected, the following six molecular properties cannot be satisfied:

0010, 0011, 0110, 0111, 0001, 0101.

We constructed in Exoo and Harary 1981 the smallest graphs satisfying each of the other ten molecular properties. Our present purpose is to construct the corresponding smallest multigraphs. For four of these ten families of multigraphs, the smallest member (Fig. 2) can be quickly found by an exhaustive search of cubic multigraphs of small order. Each of these is unique.

A theta graph is a subdivision of $K_{2,3}$. It is shown in (Harary 1969, p. 66) that every 2-connected nonhamiltonian graph contains a theta subgraph. This observation is easily extended to multigraphs and is useful in determining the smallest members in several other families.

Four more cases are settled by the following theorem. In the proof we shall employ the following notational convention. If points $u$ and $v$ of a graph $G$ are adjacent, then we write $uAv$.

*Theorem* — The smallest multigraphs satisfying 1110, 1100, 1010 and 0000 are those of Fig. 3.

PROOF: In each of the first three cases, the extremal multigraphs must contain a subdivision of $K_{2,3}$.

We first show that the first multigraph in Fig. 3 is the only cubic multigraph with order $p \leqslant 8$ which is both nonhamiltonian and 2-connected.

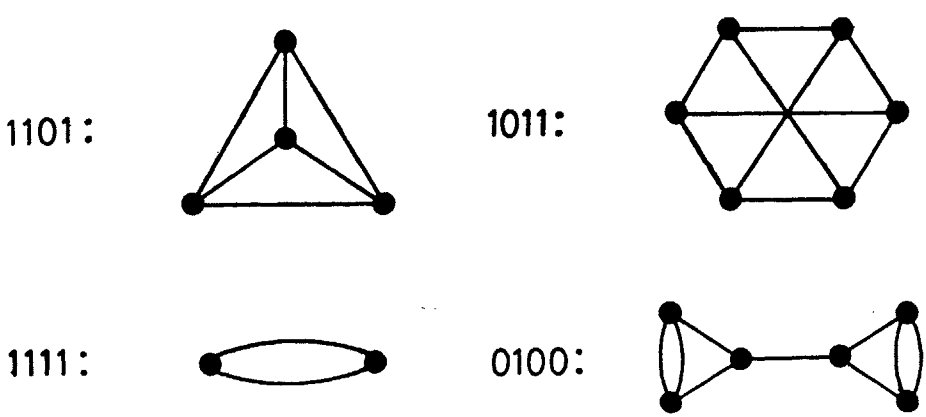


FIG. 2. Four smallest members.

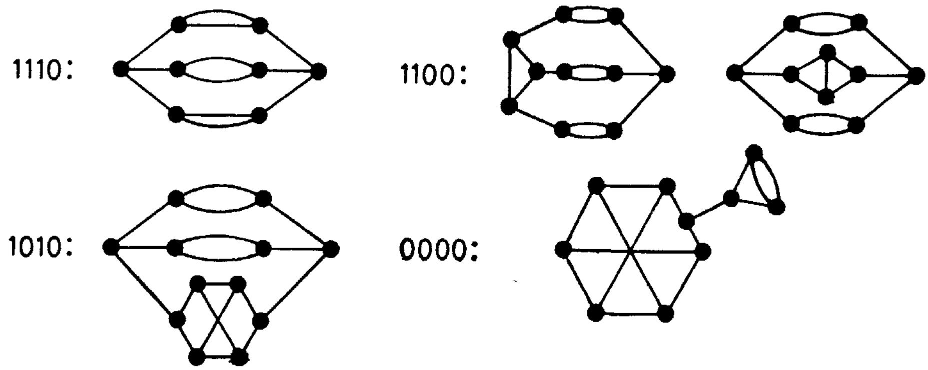


FIG. 3. More smallest numbers.

Suppose $M$ is a 2-connected nonhamiltonian cubic multigraph. If $p \leqslant 8$ it is quite easy to show that the length of a longest cycle in $M$ must be 6. This is done by a routine case by case analysis. Let $v_1, v_2, \ldots, v_6$ be the points (labelled cyclically) of the 6-cycle and let $u_1$ and $u_2$ be the remaining two points. If $u_1$ and $u_2$ are not adjacent, then each is adjacent to three points of the 6-cycle. In order to avoid cycles whose length exceeds six, we have, without loss of generality, $u_1$ adjacent to $v_1, v_3, v_5$. (using 2-connectedness). Hence $u_2Av_2v_4v_6$, whence $M$ is hamiltonian, a contradiction. Next suppose that there is one line joining $u_1$ and $u_2$. Then each is adjacent to two points at distance two on the 6-cycle. If $u_1Av_1v_3$ then $u_2$ is not adjacent to $v_2$ or else $M$ is hamiltonian. So $u_2Av_4v_6$, but again $M$ is hamiltonian.

Therefore we are left with the case that $u_1$ and $u_2$ are joined by two lines and each is adjacent to one point of the 6-cycle. Since $M$ is 2-connected $u_1$ and $u_2$ are adjacent to different points of the 6-cycle. Since $M$ is nonhamiltonian they are not adjacent to two consecutive points of the 6-cycle. If $u_1$ and $u_2$ are adjacent to two points at distance 2 on the 6-cycle, then one sees that $M$ is either not a cubic multigraph or else is hamiltonian. The only remaining possibility is that $u_1$ and $u_2$ are adjacent to points at distance three on the 6-cycle, say $v_1$ and $v_4$. Now in order that $G$ be cubic and nonhamiltonian we must also have $v_2$ and $v_3$, and $v_5$ and $v_6$ joined by 2 lines. So $M$ is the first multigraph in Fig. 3, as claimed, and uniqueness follows.

It also follows that any 1100 cubic multigraph has at least 10 points. In Fig. 3 we show the only two such multigraphs while omitting the proof which is similar.

To show that the multigraph in Fig. 3 is a smallest 1010 cubic multigraph, simply observe that any such multigraph $M$ must contain a copy of $K_{3,3}$ with at least one line subdivided. Since $M$ is bipartite, the line must be subdivided by an even number of interior points. But now we use the fact that $M$ contains a theta subgraph which implies that $M$ has order $p \geqslant 12$.

Any 0000-multigraph must contain both a cutpoint and a subdivision of $K_{3,3}$. It follows almost at once that the last multigraph of Fig. 3 is the smallest member of this family.

1001:

1000:

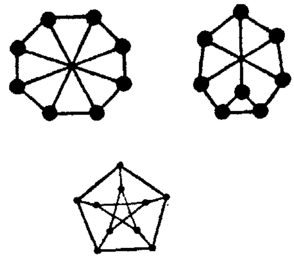

FIG. 4. The last smallest members.

There remain only two families. The extremal multigraphs for both 1001 and 1000 are in fact graphs. Shown in Fig. 4 are the möbius ladder $M_8$ defined in Guy and Harary (1967). The other smallest 1001 multigraph, and the familiar Petersen graph which is the extremal 1000 multigraph.

*Unsolved Problems*

The definitive work on extremal graph theory is the book by Bollobás (1978). Many other questions concerning extremal graphs and multigraphs suggest themselves. Each of the four atomic properties $A_i$ can be restated in terms of the value of a graphical parameter as follows:

$A_1 k \geqslant 2$ (connectivity)

$A_2 \gamma = 0$ (genus) and $\nu = 0$ (crossing number)

$A_3 \chi = 2$ (chromatic number)

$A_4 g = p$ (girth).

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