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CYCLE-VANISHING EDGE VALUATIONS OF A GRAPH

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A cycle-vanishing edge valuation of a graph G is a real-valued function on the edge set of G which vanishes on cycles of G. This paper deals with the properties of cycle-vanishing edge valuations of a graph. It also determines the dimension of the real vector space defined by these mappings. As a consequence, a characterisation of 3-edge connected graphs is obtained.

In this paper, by a graph we mean a finite graph without loops, multiple edges and isolated vertices. Let $E(G)$ denote the edgeset of the graph G.

For a subgraph H of G, and a real function f defined on $E(G)$, let $f(H) = \sum_{x \in E(H)} f(x)$. A cycle-vanishing edge valuation of a graph G is a function $f : E(G) \to R$ (where R is the field of real numbers) such that for every cycle C of G, $f(C) = 0$. A function $f : E(G) \to R$ is said to vanish identically at some vertex v of G if $f(x) = 0$ for each edge x incident with v.

PROPERTIES OF CYCLE-VANISHING EDGE VALUATIONS

Clearly, it suffices to consider cycle-vanishing edge valuations of connected graphs. Accordingly, all our graphs in this section are connected unless stated otherwise.

Lemma 1 — Let f be a cycle-vanishing edge valuation of a graph G. Let C_1 and C_2 be any two cycles of G having at least one edge of G in common. Then f vanishes on $C_1 \cap C_2$, that is,

$$f(C_1 \cap C_2) = \sum_{x \in C_1 \cap C_2} f(x) = 0.$$

PROOF : First assume that $C_1 \cap C_2$ is a path p. Thus if $C_1' = C_1 - p$ and $C_2' = C_2 - p$, then $C_1' \cup p$, $C_2' \cup p$, $C_1' \cup C_2'$ are cycles, and therefore $0 = f(C_1' \cup p) + f(C_2' \cup p) = f(C_1' \cup C_2') + 2f(p) = 2f(p)$. Thus $f(C_1 \cap C_2) = 0$.

In the general case, $C_1 - (C_1 \cap C_2)$ is a disjoint union of paths each of which is the complete intersection of two cycles of G. Hence $f(C_1 - (C_1 \cap C_2)) = 0$ and therefore $f(C_1 \cap C_2) = 0$.

Lemma 2 — If C is a cycle of any block G, there exists a sequence of non-separable subgraphs $C = B_0, B_1, \ldots, B_r = G$ such that B_{i+1} is an edge-disjoint union of B_i and a path p_i in such a way that the only vertices common to B_i and p_i are the end vertices of p_i.

PROOF : Assume that we have already determined B_i. If $B_i \neq G$, and if there exists an edge x of G which does not belong to B_i but both of its end vertices belong to B_i, take $B_{i+1} = B_i \cup x$. Otherwise take an edge $x = uv$ of G where $u \in B_i$ and $v \notin B_i$. If u' is any vertex of B_i other than u, edge x and u' lie in a common cycle C_i as G is a block. If u_i is the first vertex after u which is common to B_i and C_i as we trace the cycle C_i in the direction u to v, call the $u - u_i$ path through v as p_i. Clearly $B_i \cup p_i = B_{i+1}$ is also non-separable and the proof is now clear by induction on i.

Theorem 1 — If f is a cycle-vanishing edge valuation of a block $G \neq K_2$, then f vanishes on G.

PROOF : By Lemma 2, there exists a sequence of non-separable subgraphs $C = B_0, B_1, \ldots, B_r = G$ with respect to some cycle C. Now, by the definition of p_i, $f(p_i) = 0$ (Lemma 1). As $f(B_{i+1}) = f(B_i) + f(p_i)$ and $f(B_0) = 0$, by induction $f(B_r) = 0$, that is, $f(G) = 0$.

Lemma 3 — If G is a block with minimum degree $\delta(G) \geqslant 3$, then G has at least two vertices u and v such that $G - u$ and $G - v$ are blocks.

PROOF : By Lick (1969), G has a vertex, say u, such that $G - u$ is a block. Take a copy G' of G and attach it to G so that G and G' have u as the only common vertex. Now join a vertex w adjacent to u in G with its corresponding vertex w' in G'. This gives a block H with $\delta(H) \geqslant 3$. Hence H has a vertex v such that $H - v$ is a block. Note that v must be different from both u and w. Further v can be taken to be a vertex of G. Since $H - v$ is a block, any two distinct vertices of $H - v$ lie on a cycle. If this cycle contains an edge of G', it must necessarily contain the vertex u and the edge ww'. But then as w is adjacent to u, we get a cycle in $G - v$ containing the two vertices. In other words, $G - v$ is a block.

Lemma 4 — If G is a block with exactly one vertex w of degree 2 in G, then G has a vertex v different from w such that $G - v$ is a block.

PROOF : The proof is similar to that of Lemma 3 and is obtained by considering the graph H obtained by attaching a copy G' of G to G at the vertex w and joining a vertex adjacent to w in G to its corresponding vertex in G'.

Theorem 2 — If f is a cycle-vanishing edge valuation of a block G with $\delta(G) \geqslant 3$, then f vanishes identically at two vertices of G.

PROOF : By Lemma 3, G has at least two vertices u and v such that $G - u$ and $G - v$ are blocks. Let $uu_1 = x$, $uu_2 = y$, $uu_3 = z$ be any three edges incident at u in G. Since $G - u$ is a block, the two vertices u_1 and u_2 lie on a cycle of $G - u$ and hence the path u_1uu_2 becomes the full intersection of two different cycles of G. Hence by Lemma 1, $f(x) + f(y) = 0$. By symmetry, $f(y) + f(z) = f(z) + f(x) = 0$. Hence $f(x) = f(y) = f(z) = 0$. As x, y and z are three arbitrary edges incident at u, f vanishes identically at u and similarly at v.

SPACE OF CYCLE-VANISHING EDGE VALUATIONS

For any graph G, it is clear that the set of all cycle-vanishing edge valuations of G forms a real vector space $\mathcal{CV}_G$. We now proceed to determine the dimension of $\mathcal{CV}_G$. Let $E_0(G) = E(G) \setminus \{\text{set of bridges of } G\}$. For $x, y \in E_0(G)$, set $x \sim y$ iff any cycle of G that contains x also contains y. Then $\sim$ is an equivalence relation on $E_0(G)$. Trivially it is reflexive and transitive. To see that it is symmetric, let $x \sim y$ but $y \nsim x$. Then there exists a cycle C_2 af G containing y but not x. As x is not a bridge of G, there exists a cycle C_1 of G containing x and therefore y. But then by the properties of cycles of a graph (Welsh 1976, p. 25), there exists a cycle C_3 such that $x \in C_3 \subset (C_1 \cup C_2) \setminus y$, a contradiction.

Let $[x]$ denote the equivalence class defined by $x \in E_0 = E_0(G)$.

Theorem 3 — $f: E(G) \to R$ is a cycle-vanishing edge valuation of G iff $f([x]) = 0$ for every $x \in E_0(G)$.

PROOF : If C is any cycle of G, C is a union of equivalence classes in E_0. Hence if f vanishes on each equivalence class in E_0, f vanishes on C and is therefore a cycle-vanishing edge valuation of G.

Conversely, let f be a cycle-vanishing edge valuation of G. If $x, y \in E_0$ and $y \notin [x]$, there exists a cycle C of G such that $y \in C \subset E_0 \setminus [x]$. Hence both E_0 and $E_0 \setminus [x]$ are edge-disjoint unions of blocks none of which is K_2, and so by Theorem 1, f vanishes on each of them. Thus $f([x]) = 0$.

As a consequence of Theorem 3, a cycle-vanishing edge valuation of G is obtained by arbitrarily fixing its values on the bridges of G and on all but one edge of each of the equivalence classes in E_0. This gives

Theorem 4 — The dimension of the real vector space of all cycle-vanishing edge valuations of a graph G is $q - r$, where q is the number of edges of G and r is the number of distinct equivalence classes $[x]$, $x \in E_0(G)$.

As a consequence of Theorem 4, we get the following characterisation of 3-edge connected graphs.

Theorem 5 — A connected graph G is 3-edge connected iff the zero mapping is the only cycle-vanishing edge valuation of G.

PROOF : If the zero mapping is the only cycle-vanishing edge valuation of G dim $\mathcal{CV}_G = q - r = 0$ so that there are no bridges in G and each equivalence class is a singleton. This however means that for any two edges of G, there exists a cycle containing one but not the other. Thus G is 3-edge connected.

As for the converse, if G is 3-edge connected, and if x and y are any two edges of G, y cannot be a bridge of $G - x$. Hence y lies in a cycle of G not containing x. Consequently, $q = r$ and the zero mapping is the only cycle-vanishing edge valuation of G.

The following corollary to Theorem 5 is of practical utility in checking whether a graph is 3-edge-connected or not.

Corollary — A connected graph G is 3-edge-connected iff each edge of G is the intersection of the edge sets of two cycles of G.

PROOF : If G is 3-edge-connected and $x = uv$ is any edge of G, there exist two edge-disjoint $u - v$ paths p_1 and p_2 in $G - x$ and hence x is the intersection of the edge sets of the two cycles $p_1 \cup x$ and $p_2 \cup x$. Conversely, the condition implies, by virtue of Lemma 1, that any cycle-vanishing edge valuation of G must be the zero mapping and hence, by Theorem 5, G must be 3-edge-connected.

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