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CLASSIFICATION OF LINE COMPLEXES IN THE FLAG SPACE F_3

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In the present paper, a complete classification of line complexes in third differential neighbourhood of the ray in the flag space F_3, based on the symmetry of the matrices which are arised from the integrability conditions of the system of differential equations of a line complex, is given. Mainly two subclasses are concerned, for each one the existence theorem is proved and the integral-free representation is given. The methods adopted are based on Cartan's differential calculus (Finikov 1948).

1. INTRODUCTION

A homogeneous space $F_3 \equiv (P_3, G_6)$ is called flag space if P_3 is 3-dimensional projective space with metric and G_6 is a six-fold subgroup of the group of projective transformations of P_3 that has degenerate absolutam consists of a plane at infinity with an invariant line and invariant point on it (Penner 1967).

We choose a moving frame conjugate to any arbitrary manifold immersed in the flag space F_3 as a coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ are the points at infinity and $\bar{A}_4$ is a proper point such that the edges $\bar{A}_4\bar{A}_1, \bar{A}_4\bar{A}_2, \bar{A}_4\bar{A}_3$ form orthogonal traid. The invariant point $\bar{A}_1$ and the invariant line $\bar{A}_1\bar{A}_2$ lie on the invariant plane $\bar{A}_1\bar{A}_2\bar{A}_3$ (the ideal plane) (Redei 1968). The fundamental equations of the moving frame T are

$$d\begin{bmatrix}\bar{A}_1\\ \bar{A}_2\\ \bar{A}_3\\ \bar{A}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\\ \omega_2^1 & 0 & 0 & 0\\ \omega_3^1 & \omega_3^2 & 0 & 0\\ \omega_4^1 & \omega_4^2 & \omega_4^3 & 0\end{bmatrix}\begin{bmatrix}\bar{A}_1\\ \bar{A}_2\\ \bar{A}_3\\ \bar{A}_4\end{bmatrix} \quad \ldots(1)$$

and the structural equations (i.e., the integrability conditions) are

$$D\begin{bmatrix}\omega_2^1\\ \omega_3^1\\ \omega_3^2\\ \omega_4^1\\ \omega_4^2\\ \omega_4^3\end{bmatrix} = \begin{bmatrix}0&0&0&0&0&0\\ \omega_3^2&0&0&0&0&0\\ 0&0&0&0&0&0\\ \omega_4^2&\omega_4^3&0&0&0&0\\ 0&0&\omega_4^3&0&0&0\\ 0&0&0&0&0&0\end{bmatrix}\wedge\begin{bmatrix}\omega_2^1\\ \omega_3^1\\ \omega_3^2\\ \omega_4^1\\ \omega_4^2\\ \omega_4^3\end{bmatrix} \quad \ldots(2)$$

where ω_i^j are Pfaff's differential form, D denotes the exterior differentiation operator and $\wedge$ the exterior product between the differential forms.

The differential equations of a line complex immersed in F_3, which generated by the ray $l \equiv (\bar{A}_3\bar{A}_4)$, related to its canonical frame in the second differential neighbourhood of its ray (Soliman and Abdel-All 1980) are

$$\begin{bmatrix} \omega_4^1 \\ \omega_2^1 \\ -\omega_4^3 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ p & \alpha & \beta & 0 \\ \alpha & q & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix} \quad \ldots(3)$$

where k is an invariant of the first differential neighbourhood and is called the curvature of the line complex; p, α, β, q, γ, r are the invariants of the second differential neighbourhood of the ray.

Exterior differentiation of the Pfaffian system (3) and using Cartan's lemma, we get the following systems of equations

$$d\,[p\alpha\beta]^t = M_1\Omega, \quad d\,[\alpha q\gamma]^t = M_2\Omega, \quad d\,[\beta\gamma r]^t = M_3\Omega \quad \ldots(4)$$

where t denotes the transpose of a row matrix,

$$\Omega = \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_4^2 \end{bmatrix}, \; M_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 - \alpha p \\ \lambda_2 & \lambda_4 & \lambda_5 - \alpha^2 \\ \lambda_3 + \alpha p & \lambda_5 + pq & \lambda_6 \end{bmatrix}, \quad \ldots(5)$$

$$M_2 = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 - pq \\ \mu_2 & \mu_4 & \mu_5 - \alpha q \\ \mu_3 + \alpha^2 & \mu_5 + \alpha q & \mu_6 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 - p\gamma \\ \nu_2 & \nu_4 & \nu_5 - \alpha\gamma \\ \nu_3 + \alpha\beta & \nu_5 + \beta q & \nu_6 \end{bmatrix},$$

where $\lambda_1, \ldots, \lambda_6$; $\mu_1, \ldots, \mu_6$ and $\nu_1, \ldots, \nu_6$ are the invariants in the third differential neighbourhood of the ray.

Orehova (1970) has introduced the notation of line complex with skew symmetric matrix in Euclidean space. We give therein a complete classification of line complexes in the space F_3 for which both M_1 and M_2 are symmetrical matrices where as M_3 is not.

The matrix M_1 is symmetric if and only if the matrix M_2 is symmetric and this is possible only if one of the following conditions (a) $\alpha = q = 0$, (b) $\alpha = p = 0$, is satisfied. We denote by $(K)_1$, $(K)_2$ and $(K)_{12}$ classes of line complexes for which M_1; M_2 and M_1, M_2 are symmetrical matrices respectively.

In this classification, the classes $(K)_1$, $(K)_2$ and $(K)_{12}$ are the same, we denote them by $(K)_I$. The class $(K)_I$ contains two subclasses $^a(K)_I$, $^b(K)_I$ according to the conditions (a), (b) respectively.

2. The Subclass of Line Complexes $^a(K)_I$

Any line complex $K \in {}^b(K)_I$ characterized by the system of differential eqns. (3) with the condition $\alpha = q = 0$ and the system of differential equations

$$d\begin{bmatrix} p \\ \beta \\ \gamma \\ r \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_6 & 0 \\ 0 & 0 & \mu_6 & 0 \\ \lambda_6 & \mu_6 & \nu_6 & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix} \qquad \ldots(6)$$

in the third differential neighbourhood of the ray. For the line complex $K \in {}^a(K)_I$, $\bar{A}_4$ the centre of the ray is a centre of inflection (Kovansov 1963).

From (3) and (6), it follows that, the number of independent parameters $N = 5(\lambda_1, \lambda_3, \lambda_6, \mu_6, \nu_6)$. Using Cartan's method, we have the number of characteristic forms $\tilde{q} = 4(dp, d\beta, d\gamma, dr)$, the number of independent exterior forms $S_1 = 3$, and the equality $\tilde{q} = S_1 + S_2$ gives that $S_2 = 1$. Hence the Cartan's number $Q = S_1 + 2S_2 = 5 = N$. This means that the system of differential eqns. (6) is in involution and exists within one arbitrary function of two arguments. Then, we have the following existence theorem.

Theorem 1.1 — The range of the existence of every line complex belongs to the subclass $^a(K)_I$ comprises one arbitrary function of two arguments.

Geometrical Properties of the Subclass $^a(K)_I$

We investigate some geometrical properties of the line complexes $K \in {}^a(K)_I$, which can help us to give their integral-free representation (i.e., geometrical construction) (Veselova 1975).

Property 1.1 — The line complex $K \in {}^a(K)_I$ is stratified into one parameter family of hyperbolic holonomic line congruences $^aC_1$, $\omega_3^2 = 0$, with focal surfaces degenerate to the ideal line $\bar{A}_3\bar{A}_1$ and a plane curve which are described by the focal points $\bar{A}_3$ and $\bar{A}_4$ respectively.

Property 1.2 — The line complex $K \in {}^a(K)_I$ is stratified into one parameter family of hyperbolic holonomic line congruences $^aC_2$, $p\omega_4^2 + \beta\omega_3^2 = 0$, with a space curve and developable surface as the focal surfaces described by the focal point $\bar{A}_4$ and $\bar{A}_4 + (\beta/p)\,\bar{A}_3$ respectively.

The Integral-free Representation of the Subclass $^a(K)_I$

The meaning of integral-free representation of a line complex is to obtain its system of differential equations from its geometrical construction (Kovansov and Ponomarov 1975).

Theorem 1.2 — Considering an arbitrary surface Σ_a, the lines drawn from every point belonging to this surface will meet the ideal plane. The family of all these lines construct any line complex $K \in {}^a(K)_I$.

PROOF : We place the proper vertex $\bar{A}_4$ of the canonical coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$ on the surface Σ_a. The vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ form the ideal plane. The tangent plane of the surface Σ_a at the point $\bar{A}_4$ meet the ideal plane at the line $\bar{A}_2\bar{P}$ and $\bar{P}$ is a linear combination of the points $\bar{A}_1, \bar{A}_3$.

Therefore, from this construction, the ray $\bar{A}_4\bar{A}_3$ coincident with the ray of the line complex which is determined from the system of eqns. (3).

Since the point $\bar{A}_4$ describes the surface Σ_a with tangent plane $\bar{A}_4\bar{A}_2\bar{P}$, then $d\bar{A}_4 = m\bar{P} + \omega_4^2 \bar{A}_2$. Comparing the differential $d\bar{A}_4$ with the eqn. (1), it follows that, $\omega_4^1 \wedge \omega_4^3 = 0$, i.e., ω_4^3 is a function of ω_4^1 only. From this result and the system of eqns. (3), we have $\alpha = q = 0$, which characterize the constructed line complex of the subclass $^a(K)_I$.

3. THE SUBCLASS OF LINE COMPLEXES $^b(K)_I$

Any line complex $K \in {}^b(K)_I$ characterized by the system of differential eqns. (3) with the condition $\alpha = p = 0$, and the system of differential equations

$$d\begin{bmatrix} \beta \\ \gamma \\ q \\ r \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_6 & 0 \\ 0 & \mu_5 & \mu_6 & 0 \\ 0 & \mu_4 & \mu_5 & 0 \\ \lambda_6 & \mu_6 + \beta q & \nu_6 & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix} \qquad \ldots(7)$$

in the third differential neighbourhood of the ray. For the line complex $K \in {}^b(K)_I$, $\bar{A}_3$ the centre of the ray is a centre of inflection. From eqns. (3) and (7), it is easy to prove the following existence theorem by similar way as that mentioned before.

Theorem 2.1 — The range of existence of the complexes belongs to the subclass $^b(K)_I$ comprises one arbitrary function of two arguments.

Geometrical Properties of the Subclass $^b(K)_I$

Property 2.1 — Every line complex $K \in {}^b(K)_I$ admitting a fibration into one parameter family of a hyperbolic holonomic congruence $\omega_3^2 = 0$, with developable surface and ideal line $\bar{A}_3\bar{A}_1$ as focal surfaces described by the distinct focal points $\bar{A}_4$ and $\bar{A}_3$ respectively.

Property 2.2 — If the ray $\bar{A}_4\bar{A}_3$ generates the line complex $K \in {}^b(K)_l$, the ray $\bar{A}_4\bar{A}_2$ generates a hyperbolic line congruence C. The focal surfaces of C are the invariant line $\bar{A}_2\bar{A}_1$ and arbitrary surface with tangent plane $\bar{A}_4\bar{A}_2\bar{A}_3$ which are described by the focal points $\bar{A}_2$ and $\bar{A}_4 - (k/\beta)\,\bar{A}_2$ respectively.

Property 2.3 — If the centre of the ray $\bar{A}_3$ is stable, then the centre of the ray $\bar{A}_4$ moves along the generator $\bar{A}_4\bar{A}_2$ of the congruence C. i.e., the ray $\bar{A}_4\bar{A}_3$ generates a bundle of lines with vertex $\bar{A}_3$ on the ideal plane and $\bar{A}_4\bar{A}_2$ as a layer line.

The Integral-free Representation of the Subclass ${}^b(K)_l$

From the foregoing properties of the line complex $K \in {}^b(K)_l$, we have the theorem which gives the integral-free representation of this subclass.

Theorem 2.2 — We consider an arbitrary surface Σ_b and the invariant line on the ideal plane as the focal surfaces of a line congruence C. For each generator m of the line congruence C, construct a bundle of lines with layer on m and vertex on the ideal plane. All these bundles represents a line complex of the subclass ${}^b(K)_l$.

PROOF : On a generator m of the line congruence C, we place the vertices $\bar{A}_4$ and $\bar{A}_2$ of a canonical coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, the generator m intersects the ideal plane at the point $\bar{A}_2$, also the vertices $\bar{A}_1$ and $\bar{A}_3$ placed on the ideal plane and the focal points of the congruence coincident with $\bar{F} = \bar{A}_4 + \tau\bar{A}_2$, $\bar{A}_2$. Since every line complex $K \in {}^b(K)_l$ will be constructed as the family of all bundles of lines with vertices on the ideal plane and layers coincident with the generators of the line congruence C, hence the ray $\bar{A}_4\bar{A}_3$ generates the line complex $K \in {}^b(K)_l$. Any arbitrary line complex immersed in F_3 characterized by the system of eqns. (3).

The point $\bar{F} = \bar{A}_4 + \tau\bar{A}_2$ is a focal point of the congruence C if $d\bar{F} \equiv O \,(\text{mod}\, \bar{F})$, which gives $\omega_4^1 + \tau\omega_2^1 = 0$, $(1/\tau)\,\omega_4^3 = 0$. The value $\tau = \infty$ corresponds to the focal point $\bar{A}_2$. For the orther value of τ, the forms ω_4^1, ω_2^1, must be satisfy $\omega_4^1 \wedge \omega_2^1 = 0$, i.e., ω_2^1 is a function of ω_4^1 only. From this result and using the system of eqns. (3) we have $\alpha = p = 0$, which characterized the construted line complexes of the subclass ${}^b(K)_l$.

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