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A CLASSIFICATION OF LINE COMPLEXES IMMERSED IN QUASI-HYPERBOLIC SPACE "S_3' THAT BASED ON THE MOBILITY OF A REPERE

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The object of this paper is to obtain some geometrical properties of certain classes of line complexes in the first differential neighbourhood. This has been done by a complete classification of line complexes that is based on the mobility of a repere which conjugate to any arbitrary manifold imbedded in "S_3'. The methods adapted are based on Cartan's differential calculus (Finkov 1948).

INTRODUCTION

A homogeneous space "S_3" $\equiv (P_3, G_6)$ is called quasi-hyperbolic space if P_3 is a 3-dimensional projective space with projective metric and G_6 is a subgroup of the group of projective transformations of P_3 which preserves two real points belonging to two intersecting planes (Abdurahmanova 1977). This means that, we consider the geometry belonging to the following six-fold group G_6 of projective transformations

$$\begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \\ x'^4 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ a_{21} & 1 & 0 & a_{24} \\ a_{31} & 0 & 1 & a_{34} \\ a_{14} & 0 & 0 & a_{11} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} \cdot (a_{11})^2 \cdot (a_{14})^2 \neq 0.$$

We construct a repere mobile conjugate to any arbitrary manifold immersed in "S_3' as the coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_2, \bar{A}_3$ are the two invariant points and the two invariant planes are $(\bar{A}_2\bar{A}_3, \bar{A}_1 + \bar{A}_4)$, $(\bar{A}_2\bar{A}_3, \bar{A}_1 - \bar{A}_4)$. Then the absolutum in the mentioned repere mobile consists of the two intersecting planes $x^1 \pm x^4 = 0$ and the two real points (0, 1, 0, 0), (0, 0, 1, 0) in homogeneous coordinates (Rosenfeld 1969).

The fundamental equations of the repere mobile T are

$$\begin{bmatrix} d\bar{A}_1 \\ d\bar{A}_2 \\ d\bar{A}_3 \\ d\bar{A}_4 \end{bmatrix} = \begin{bmatrix} \omega_1^1 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ 0 & -2\omega_1^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_1^4 & \omega_4^2 & \omega_4^3 & \omega_1^1 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} \qquad \ldots(1)$$

and the structural equations (i.e., the integrability conditions) are

$$\begin{bmatrix} D\omega_1^1 \\ D\omega_1^2 \\ D\omega_1^3 \\ D\omega_1^4 \\ D\omega_4^2 \\ D\omega_4^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\omega_1^1 & 0 & 0 & \omega_1^4 & 0 \\ 0 & 0 & \omega_1^1 & 0 & 0 & \omega_1^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1^4 & 0 & 0 & 3\omega_1^1 & 0 \\ 0 & 0 & \omega_4^1 & 0 & 0 & \omega_1^1 \end{bmatrix} \wedge \begin{bmatrix} \omega_1^1 \\ \omega_1^2 \\ \omega_1^3 \\ \omega_1^4 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix} \quad \ldots(2)$$

where ω_i^j are Pfaff's differential forms, D denotes the exterior differentiation operator and $\wedge$ the exterior product between the differential forms. In the repere mobile T, the Pfaff's equation $\omega_1^4 = 0$ is completely integrable which determines the two intersecting coordinate planes $x^4 = 0$ and $x^1 = 0$, described by the vertices $\bar{A}_1$ and $\bar{A}_4$ respectively. The repere mobile T has the property that, the invariant planes $x^1 \pm x^4 = 0$ and the coordinate planes $x^1 = 0$, $x^4 = 0$ will form a harmonic set.

THREE-DIMENSIONAL LINE MANIFOLDS IMBEDDED IN "S_3"

Let, in S_3, u, v, w, be a system of curvilinear coordinates; the admissible variables (u, v, w) are taken from an open neighbourhood of C^3. Consider a 3-dimensional line manifold (line complex) imbedded in S_3, which generated by the ray $l = (\bar{A}_1\bar{A}_4)$ where $l = l(u, v, w)$ is Klein point and $(\bar{A}_1\bar{A}_4)$ is the Grassmann coordinates of the line $\bar{A}_1\bar{A}_4$. From the differential

$$dl \equiv \omega_1^2\,(\bar{A}_2\bar{A}_4) + \omega_1^3\,(\bar{A}_3\bar{A}_4) + \omega_4^2\,(\bar{A}_1\bar{A}_2) + \omega_4^3\,(\bar{A}_1\bar{A}_3)\ \{\mathrm{mod}\ l\}$$

it follows that, the differential equation of a line complex related to the repere mobile T in the first differential neighbourhood of its ray {first order contact elements of the line complex} is given by

$$\omega_1^2 = a\omega_1^3 + b\omega_4^2 + c\omega_4^3 \quad \ldots(3)$$

where ω_i^j are linear functions in the differentials du, dv, dw and ω_1^3, ω_4^2, ω_4^3 are linearly independent Pfaffian forms which determine a line complex immersed in S_3. The parameters a, b and c are differentiable functions in u, v, w and they are called invariants of the first order contact elements of l (Borisov 1971).

The variation in the invariants a, b and c is

$$\begin{bmatrix} \delta a \\ \delta b \\ \delta c \end{bmatrix} = \begin{bmatrix} 2a & -c-ab \\ 0 & 1-b^2 \\ 2c & -a-cb \end{bmatrix} \begin{bmatrix} \pi_1^1 \\ \pi_1^4 \end{bmatrix} = M\,[\pi_1^1\ \pi_1^4]^t \quad \ldots(4)$$

where t denotes the transpose of a row matrix

$$M = \begin{bmatrix} 2a & -c - ab \\ 0 & 1 - b^2 \\ 2c & -a - cb \end{bmatrix},$$

δ denotes the differentiation such that $\delta u = \delta v = \delta w = 0$ and π_i^j is the differential form arising form ω_i^j by putting $du = dv = dw = 0$ in it. In general, the matrix M has rank equal two.

We give therein a complete classification of line complexes (3) that is based on the mobility of the repere T, for which the rectangular matrix M has a rank lower than two (Ponomarov 1972).

1. The First Class of Line Complexes (K_1)

For this class of line complexes, the rank of the matrix M equal to zero which gives $a = c = 0$, $b = \pm 1$ and from the eqn. (3), it follows that any line complex $K \in (K)_1$ will be defined by the completely integrable Pfaffian equation

$$\omega_1^2 \pm \omega_4^2 = 0 \qquad ...(5)$$

Hence, we have the following existence theorem.

Theorem 1.1 — The range of the existence of a line complex $K \in (K)_1$ comprises only one arbitrary constant.

Geometrical Properties of the Class $(K)_1$

Property 1.1 — The plane corresponding to any point $\bar{F} = \bar{A}_1 + t\bar{A}_4$ on the ray l in the normal correlation coincident with the coordinate plane $x^2 = 0$. In other words, the cone of rays of the line complex $K \in (K)_1$ will be a cylindrical surface.

Property 1.2 — The Pfaffian system of equations

$$\omega_4^2 = 0,\ \omega_1^2 = 0,\ \omega_1^3 + \omega_4^3 = 0 \qquad ...(6)$$

determine a pencil of lines (one-dimensional line manifold) with vertex on the invariant plane.

PROOF : From $D\omega_4^2 \equiv 0 \{\text{mod } \omega_4^2\}$, $D\omega_1^2 \equiv 0 \{\text{mod } \omega_1^2\}$,

$$D\{\omega_1^3 + \omega_4^3\} \equiv 0 \{\text{mod } \omega_1^3 + \omega_4^3\} \text{ and } dl \equiv \omega_4^3 (\bar{A}_1, + \bar{A}_4, \bar{A}_3) \{\text{mod } l\},$$

it follows that, the Pfaffian system (6) is completely integrable and determine a ruled surface Σ with stable contiguous plane $\sigma = (\bar{A}_1\bar{A}_4\bar{A}_3)$. The point $\bar{P}_1 = \bar{A}_1 + \bar{A}_4$ on the ray l is a fixed point. Finally we conclude that, the ruled surface Σ is degenerate to a pencil of lines in the plane σ with vertex $\bar{P}_1$ on the invariant plane $x^1 + x^4 = 0$.

Corollary 1.1 — The line complex $K \in (K)_1$ admits a stratification into two-parameter families of the pencils of lines (6) (Kovancov 1963).

Property 1.3 — The complete integrable Pfaffian system of equations $\omega_4^2 = 0$, $\omega_1^2 = 0$, $dt + (1 - t^2)\,\omega_1^4 = 0$, gives that, any analytic point $\bar{F}$ on a ray of the line complex $K \in (K)_1$ describes a pencil of lines with vertex on the absolutum. This pencil of lines will orthogonally cut the rays of the line complex $K \in (K)_1$.

Property 1.4 — The complete integrable Pfaffian equations

$$\omega_4^2 = 0,\ \omega_1^2 = 0 \qquad \ldots(7)$$

determine a bundle of lines (one-dimensional line manifold) in the coordinate plane σ.

PROOF : From equations (1) and (10) we have

$$d\bar{A}_1 \equiv \omega_1^3\,\bar{A}_3 + \omega_1^4\,\bar{A}_4\,\{\text{mod}\ \bar{A}_1\},\ d\bar{A}_4 \equiv \omega_4^3\,\bar{A}_3 + \omega_4^1\,\bar{A}_1\,\{\text{mod}\ \bar{A}_4\}$$

$$dl \equiv (\omega_4^3\,\bar{A}_1 - \omega_1^3\,\bar{A}_4,\ \bar{A}_3)\,\{\text{mod}\ l\},\ d\bar{F} \equiv \{\omega_1^3 + t\omega_4^3\}\,\bar{A}_3$$

$$+ \{dt + (1 - t^2)\,\omega_1^4\}\,\bar{A}_4\,\{\text{mod}\ \bar{F}\}.$$

From these differentials, it follows that, any point $\bar{F}$ on the ray l describes the fixed plane σ and l generate a bundle of lines in the plane σ. Hence, every line complex $K \in (K)_1$ represents as a two-dimensional set of lines in the plane σ.

2. The Second Class of Line Complexes $(K)_2$

For this class of line complexes, the rank of the matrix M equal to one which gives $c = \pm a$, $b = \pm 1$ and from the eqn. (3), we see that, any line complex $K \in (K)_2$ will be characterized by the differential equation

$$\omega_1^2 = a(\omega_1^3 \pm \omega_4^3) \pm \omega_4^2. \qquad \ldots(8)$$

Theorem 2.1 — The range of the existence of any line complex $K \in (K)_2$ comprises one arbitrary function of one argument.

PROOF : Exterior differentiation of (8) and using Cartan's lemma yields that

$$da = \lambda(\omega_1^3 \pm \omega_4^3) - 2a(\omega_1^4 - \omega_1^1) \qquad \ldots(9)$$

where λ is a differentiable function which is called the invariant in the second order contact elements of the line complex. From (9), it follows that, the number of independent parameters $N = 1$ which is equal to the Cartan's number $Q = 1 = S_1$(S_1 is the number of independent quadratic exterior forms) and this verifies the existence Theorem 2.1.

Geometrical Properties of the Class $(K)_2$

Property 2.1 — The plane corresponding to any point $\bar{F}$ on the ray l in the normal correlation has the equation $(1 + t)\,x^2 - a(1 - t)\,x^3 = 0$. This plane intersects the invariant line $\bar{A}_2\bar{A}_3$ in the fixed point $\bar{F}_1 = \bar{A}_2 - \{a(1 - t)/(1 + t)\}\,\bar{A}_3$.

Property 2.2 — The complete integrable Pfaffian equations

$$\omega_1^3 + \omega_4^3 = 0,\ \omega_1^2 - \omega_4^2 = 0 \qquad \ldots(10)$$

determine a hyperbolic holonomic line congruence (two-dimensional line manifold) with two focal surfaces are coincident with the absolutum.

PROOF : From (1) and (10) we have the differential

$$dl \equiv \omega_1^2\,(\bar{A}_2, \bar{A}_1 - \bar{A}_4) + \omega_1^3\,(\bar{A}_3, \bar{A}_1 + \bar{A}_4)\ \{\text{mod } l\}$$

which depends only on two principal forms. Therefore the system of eqns. (10) determines a holonomic line congruence belongs to the line complex (8). To find the focal points of that line congruence, let $\bar{F}$ be any analytic point on the ray l. The point $\bar{F}$ is a focal point if $d\bar{F} \equiv 0$ {mod $\bar{F}$}. Hence, the focal points are the two distinct points $\bar{A}_1 \pm \bar{A}_4$. The two focal surfaces are degenerate to the invariant planes $x^1 \pm x^4 = 0$ of the absolutum.

Corollary 2.1 — Every line complex $K \in (K)_2$ admits a fibration into one-parameter families of the hyperbolic line congruences (10).

Property 2.3 — The developable surfaces of the line congruence (10) consist of two distinct systems of bundles of lines in the plane σ.

PROOF : The developables of the line congruence (10) are given by

$$(\bar{A}_1, d\bar{A}_1, \bar{A}_4, d\bar{A}_4) = \omega_4^2\,\omega_4^3 = 0. \qquad \ldots(11)$$

From the eqns. (10) and (11) we have two systems of developable surfaces defined by the two completely integrable Pfaffian equations

$$\omega_4^2 = 0,\ \omega_1^2 = 0,\ \omega_1^3 + \omega_4^3 = 0 \qquad \ldots(12)$$

$$\omega_4^3 = 0,\ \omega_1^2 - \omega_4^2 = 0,\ \omega_1^3 = 0. \qquad \ldots(13)$$

It is easy to prove that, the eqns. (12) and (13) determine two bundles of lines in the plane σ with vertices at $\bar{A}_1 + \bar{A}_4$ and $\bar{A}_1 - \bar{A}_4$ respectively. The systems of these bundles of lines (12) and (13) are the only developable systems in the line congruence (10).

The edge of regression of each bundle is a point, and the locus of this point is a curve C_f which is the focal envelope of the congruence (10). Hence, we can construct geometrically the line congruence (10) as a one-parameter set of bundles of lines with vertices at each point of the curve C_f (Hlavaty 1953).

Property 2.4 — The complete integrable Pfaffian equations

$$\omega_1^3 = 0,\ \omega_4^2 = 0,\ \omega_1^4 = 0 \qquad \ldots(14)$$

defines a hyperbolic linear line congruence with the two lines $\bar{A}_1\bar{A}_2$ and $\bar{A}_4\bar{A}_3$ as the directrices described by the focal points $\bar{A}_1$ and $\bar{A}_4$ respectively.

Corollary 2.2 — Any line complex $K \in (K)_2$ can be stratified into one-parameter families of line congruences (14).

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