Indian J. pure appl. Math., 13(3) : 329–333, March 1982

THREE-DIMENSIONAL LINE MANIFOLDS WITH A SYMMETRIC MATRIX

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(*Received* 27 *April* 1981; *after revision* 23 *November* 1981)

In the present article, a class of three-dimensional line manifold (line complex) with a symmetric matrix which arised from the integrability conditions of the Pfaffian system of equations of a line complex immersed in the flag space F_3, is concerned. For this class, the existence theorem is proved and the integral-free representation is given. The methods adopted are based on Cartan's differential calculus (Finikov 1948).

INTRODUCTION

A homogeneous space $F_3 \equiv (P_3, G_6)$ is called flag space if P_3 is three-dimensional projective space with metric and G_6 is a six-fold subgroup of the group of projective transformations of P_3 that has degenerate absolutum consists of a plane at infinity with an invariant line and invariant point on it (Penner 1967).

We choose a moving frame conjugate to any arbitrary manifold immersed in the flag space F_3 as a coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ are the points at infinity and $\bar{A}_4$ is a proper point such that the edges $\bar{A}_4\bar{A}_1, \bar{A}_4\bar{A}_2, \bar{A}_4\bar{A}_3$ form orthogonal traid. The invariant point $\bar{A}_1$ and the invariant line $\bar{A}_1\bar{A}_2$ lie on the invariant plane $\bar{A}_1\bar{A}_2\bar{A}_3$ (the ideal plane) (Redei 1968).

The fundamental equations of the moving frame T are

$$\begin{bmatrix} d\bar{A}_1 \\ d\bar{A}_2 \\ d\bar{A}_3 \\ d\bar{A}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^1 & 0 & 0 & 0 \\ \omega_3^1 & \omega_3^2 & 0 & 0 \\ \omega_4^1 & \omega_4^2 & \omega_4^3 & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} \qquad \ldots(1)$$

and the structural equations (i.e., the integrability conditions) are

$$\begin{bmatrix} D\omega_2^1 \\ D\omega_3^1 \\ D\omega_3^2 \\ D\omega_4^1 \\ D\omega_4^2 \\ D\omega_4^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_3^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_4^2 & \omega_4^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_4^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} \omega_2^1 \\ \omega_3^1 \\ \omega_3^2 \\ \omega_4^1 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix} \qquad \ldots(2)$$

where ω_i^j are Pfaff's differential form, D denotes the exterior differentiation operator and $\wedge$ the exterior product between the differential forms.

Any three-dimensional line manifold (line complex) embedded in F_3, which generated by the ray $l = (\overline{A_3 A_4})$ {l is Klein point belongs to the Klein five-dimensional projective space} related to its canonical frame in the second differential neighbourhood of its ray (Soliman and Abdel-All 1980) will be determined by the Pfaffian system of equations

$$\begin{bmatrix} \omega_4^1 \\ \omega_2^1 \\ -\omega_4^3 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ p & \alpha & \beta & 0 \\ \alpha & q & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix} \qquad \ldots(3)$$

where k is an invariant of the first differential neighbourhood (first order contact elements) and is called the curvature of the line complex and $p, \alpha, \beta, q, \gamma, r$ are the invariants of the second differential neighbourhood of the ray (second order contact elements).

Exterior differentiation of the Pfaffian system (3) and using Cartan's lemma, we get the following systems of equations.

$$\left.\begin{aligned} d\,[p \quad \alpha \quad \beta]^t &= M_1\Omega \\ d\,[\alpha \quad q \quad \gamma]^t &= M_2\Omega \\ d\,[\beta \quad \gamma \quad r]^t &= M_3\Omega \end{aligned}\right\} \qquad \ldots(4)$$

which are the conditions of integrability of the Pfaffian system (3), where t denotes the transpose of a row matrix,

$$\Omega = \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 - \alpha p \\ \lambda_2 & \lambda_4 & \lambda_5 - \alpha^2 \\ \lambda_3 + \alpha p & \lambda_5 + pq & \lambda_6 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 - pq \\ \mu_2 & \mu_4 & \mu_5 - \alpha q \\ \mu_3 + \alpha^2 & \mu_5 + \alpha q & \mu_6 \end{bmatrix}, \quad M_3 = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 - p\gamma \\ \nu_2 & \nu_4 & \nu_5 - \alpha\gamma \\ \nu_3 + \alpha\beta & \nu_5 + \beta q & \nu_6 \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_6$; $\mu_1, \ldots, \mu_6$ and $\nu_1, \ldots, \nu_6$ are the invariants in the third differential neighbourhood of the ray (third order contact elements).

Oreheva (1970) has introduced the notation of line complex with skew symmetric matrix in Euclidean space (Orehova 1970). A complete classification of line complexes in the space F_3 for which both M_1 and M_2 are symmetrical matrices where as M_3 is not have been studied in Soliman and Abdel-All (1981). We study therein a class of line complexes in the space F_3 for which M_3 is a symmetric matrix whereas M_1 and M_2 are not. We denote this class by $(K)_3$.

The matrix M_3 is symmetric if and only if

$$p/\alpha = \alpha/q = -\beta/\gamma. \qquad ...(5)$$

Theorem 1 — The range of the existence of every line complex $K \in (K)_3$ comprises one arbitrary function of two arguments.

PROOF : Any line complex $K \in (K)_3$ will be characterized by the Pfaffian system of eqns. (3) with the conditions (5) and the system of differential equations

$$\begin{bmatrix} dp \\ d\alpha \\ dq \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 - \alpha p \\ \lambda_2 & \lambda_4 & \lambda_5 - \alpha^2 \\ \lambda_4 & \mu_4 & \mu_5 - \alpha q \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix} \qquad ...(6)$$

$$\begin{bmatrix} d\beta \\ d\gamma \\ dr \end{bmatrix} = \begin{bmatrix} \lambda_3 + \alpha p & \lambda_5 + pq & \lambda_6 \\ \lambda_5 + pq & \mu_5 + \alpha q & \mu_6 \\ \lambda_6 & \mu_6 & \nu_6 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix} \qquad ...(7)$$

in third order contact elements of the ray.

Differentiating (5), using the differentials dp, $d\alpha$, dq, $d\beta$, $d\gamma$, dr and equating to zero the coefficients of the principal forms ω_4^2, ω_3^1, ω_3^2 we get

$$\left.\begin{aligned} &\gamma\lambda_1 + \beta\lambda_2 + \alpha\lambda_3 = -p\lambda_5 - p(\alpha^2 + pq) \\ &\gamma\lambda_2 = -\beta\lambda_4 - \alpha\lambda_5 - p\mu_5 - 2\alpha pq \\ &\gamma\lambda_3 + \alpha\lambda_6 = -\beta\lambda_5 - p\mu_6 \\ &\gamma\lambda_2 + q\lambda_3 = -\alpha\lambda_5 - \beta\lambda_4 - 2\alpha pq \\ &q\lambda_6 = -\gamma\lambda_5 - \beta\mu_5 - \alpha\mu_6 \\ &q(\alpha^2 + pq) = q\lambda_5 + \gamma\lambda_4 + \beta\mu_4 + \alpha\mu_5. \end{aligned}\right\} \qquad ...(8)$$

Using Cartan's common method, we obtain the further eqns. (6) and (7) together with (3) constitute a system which is closed with respect to exterior differentiation. This system determines the class of line complexes, with one arbitrary function of two arguments.

The Integral-free Representation of the Class $(K)_3$

The meaning of integral-free representation of a line complex is to obtain its system of differential equations from its geometrical construction Kovansov and Ponomarov (1975). We investigate the geometrical properties which can help us to give the geometrical construction of the class $(K)_3$.

Lemma — The complete integrable Pfaffian equation

$$\omega_3^2 = 0 \quad \ldots(9)$$

determines a hyperbolic holonomic line congruence, with a fixed line on the ideal plane and a cylindrical surface as the focal surfaces described by the focal points $\bar{A}_3$ and $\bar{A}_4$ respectively.

PROOF : From (9) and (1), we have the differential $d(\bar{A}_3\bar{A}_4) = \omega_4^2\ (\bar{A}_3\bar{A}_2) + \omega_3^1\ (\bar{A}_1\bar{A}_4)$ $\{(\bar{A}_3\bar{A}_4)$ denotes the Grassmann coordinate of the line $\bar{A}_3\bar{A}_4\}$, which depends only on two principal forms, i.e., the ray $\bar{A}_3\bar{A}_4$ generates a holonomic line congruence belonging to the line complex $K \in (K)_3$ (Kovansov 1963).

To find the focal points of the line congruence (9), let $F = \bar{A}_4 + t\bar{A}_3$ be any point on the ray $\bar{A}_3\bar{A}_4$. This point will be a focal point if $dF = O$ (mod F) $\Rightarrow$ the focal points are A_3 and A_4.

From the Pfaffian system of equations (3) with the conditions (5) we have the differentials

$$\left.\begin{aligned} &d(\bar{A}_3\bar{A}_1) = 0, \qquad d\bar{A}_4 = \omega_4^2(\bar{A}_2 - \alpha\bar{A}_3) - q\omega_3^1\ \bar{A}_3 \\ &d(\bar{A}_4\bar{A}_2) = \omega_4^2\ \{(\bar{A}_3\bar{A}_2) + (\beta/\gamma)\ (\bar{A}_4\bar{A}_1)\},\ \omega_3^2 = 0 \end{aligned}\right\}. \quad \ldots(10)$$

From (10) it follows that, the focal point $\bar{A}_3$ describes the fixed line $\bar{A}_3\bar{A}_1$ (degenerate focal surface) and the focal point $\bar{A}_4$ describes a ruled surface Σ with generator $\bar{A}_4\bar{A}_2$ $\{d(\bar{A}_4\bar{A}_2)$ depends only on one differential form$\}$. Since the generator $\bar{A}_4\bar{A}_2$ moves such that it cuts the ideal line $\bar{A}_2\bar{A}_1$, then the ruled surface Σ (the second focal surface) is a cylindrical surface.

Corollary — The line complex $K \in (K)$ admits a stratification into one-parameter families of the hyperbolic line congruences (9) (two dimensional line manifolds).

From the foregoing results, it follows that the ray $\bar{A}_3\bar{A}_4$ of the line congruence (9) will be tangent to the edge of regression of the cylindrical surface Σ (exists within one arbitrary function of one variable). Finally we conclude that the geometrical construction of the line complex $K \in (K)_3$ is based on giving one-parameter family of cylindrical surfaces. Thus, all the differential equations determining the class of line complexes $(K)_3$ are satisfied, and we have at the same time obtained a geometric interpretation of the arbitrarines of the solutions. Then we have the following theorem.

Theorems 2 — Take a one-parameter family of cylindrical surfaces. To each edge of regression C_e of a cylindrical surface, we construct a hyperbolic line congruence with a fixed line on the ideal plane and the curve C_e as the focal surfaces. All these line congruences construct the line complex $K \in (K)_3$.

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