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A CLASSIFICATION OF LINE COMPLEXES THAT IS BASED ON THREE SYMMETRICAL MATRICES

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In continuation of our work on the classification of line complexes immersed in the flag space F_3 (Soliman and Abdel-All 1982a, b), the title line complexes are investigated. Mainly three subclasses of such complexes have been obtained, for each one the existence theorem is proved and an integral-free representation is given. The methods adopted here are based on Cartan's exterior differential forms and the moving frame methods (Finkov 1948, Wladyslaw 1970).

1. INTRODUCTION

A homogeneous space $F_3 \equiv (P_3, G_6)$ is called flag space if P_3 is a three-dimensional projective space with a metric and G_6 is a six-fold subgroup of the group of projective transformations of P_3 that has a degenerate absolutum consists of the ideal plane (the plane at infinity) with an invariant line and invariant point on it. This space has the properties of the affine and therefore also of the projective geometry (Penner 1967).

We construct a repere mobile conjugate to any arbitrary manifold immersed in the flag space F_3 as the coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ are the points at infinity and $\bar{A}_4$ is proper point such that the edges $\bar{A}_4\bar{A}_1$, $\bar{A}_4\bar{A}_2$, $\bar{A}_4\bar{A}_3$ form an orthogonal traid. The invariant point $\bar{A}_1$ and the invariant line $\rho = \bar{A}_1\bar{A}_2$ lie on the invariant plane $\bar{A}_1\bar{A}_2\bar{A}_3$ (the ideal plane) (Redei 1968).

The fundamental equations of the repere mobile T are

$$\left.\begin{aligned} &d\bar{A}_1 = 0,\ d\bar{A}_2 = \omega_2^1\,\bar{A}_1,\ d\bar{A}_3 = \omega_3^1\,\bar{A}_1 + \omega_3^2\,\bar{A}_2 \\ &d\bar{A}_4 = \omega_4^1\,\bar{A}_1 + \omega_4^2\,\bar{A}_2 + \omega_4^3\,\bar{A}_3 \end{aligned}\right\} \quad \ldots(1)$$

and the structural equations (i.e., the integrability conditions) of the partial differential eqns. (1) are

$$\left.\begin{aligned} &D\omega_2^1 = D\omega_3^2 = D\omega_4^3 = 0, \quad D\omega_3^1 = \omega_3^2 \wedge \omega_2^1 \\ &D\omega_4^2 = \omega_4^3 \wedge \omega_3^2 \qquad\qquad\ \ , \quad D\omega_4^1 = \omega_4^2 \wedge \omega_2^1 + \omega_4^3 \wedge \omega_3^1 \end{aligned}\right\} \quad \ldots(2)$$

where ω_i^j are Pfaff's differential form, D denotes the exterior differentiation operator and $\wedge$ the exterior product between the differential forms.

Any three-dimensional line manifold (line complex) emmbeded in F_3, which generated by the ray $l = (\overline{A_3 A_4})$ {l is Klein point belongs to the Klein five-dimensional projective space} related to its canonical repere mobile in the second differential neighbourhood (second order contact elements) of its ray (Soliman and Abdel-All 1981) is determined by the Pfaffian system of equations

$$\begin{bmatrix} \omega_4^1 \\ \omega_2^1 \\ -\omega_4^3 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ p & \alpha & \beta & 0 \\ \alpha & q & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix}, \quad \ldots(3)$$

where k is an invariant of the first order contact elements and is called the curvature of the line complex; p, α, β, q, γ, r are the invariants of the second order contact elements of the ray.

This line complex is considered as three-dimensional submanifold of the Grassmann manifold $Gr(1, 3)$ {the manifold of all lines of the real projective space P_3}.

Exterior differentiation of the Pfaffian system of eqns. (3) and using Cartan's lemma, we get the following systems of equations

$$| dp \; d\alpha \; d\beta |^t = M_1 \Omega, \; | d\alpha \; dq \; d\gamma |^t = M_2 \Omega, \; | d\beta \; d\gamma \; dr |^t = M_3 \Omega \quad \ldots(4)$$

which are the conditions of integrability of the Pfaffian system of eqns. (3), where t denotes the transpose of a row matrix,

$$\Omega = \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix}, \; M_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 - \alpha p \\ \lambda_2 & \lambda_4 & \lambda_5 - \alpha^2 \\ \lambda_3 + \alpha p & \lambda_5 + pq & \lambda_6 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 - pq \\ \mu_2 & \mu_4 & \mu_5 - \alpha q \\ \mu_3 + \alpha^2 & \mu_5 + \alpha q & \mu_6 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 - p\gamma \\ \nu_2 & \nu_4 & \nu_5 - \alpha\gamma \\ \nu_3 + \alpha\beta & \nu_5 + \beta q & \nu_6 \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_6$; $\mu_1, \ldots, \mu_6$ and $\nu_1, \ldots, \nu_6$ are the invariants in the third order contact elements of the ray.

Orehova (1970) has introduced the notation of line complex with skew symmetric matrix in Euclidean space. A complete classification of line complexes in the space F_3 for which both M_1 and M_2 are symmetrical matrices whereas M_3 is not as well as those with M_3 is the symmetrical matrix and M_1, M_2 are not, have been studied in Soliman and Abdel-All (1982a, b).

Therein, the class of line complexes with symmetrical matrices M_1, M_2 and M_3 is classified in three separate subclasses. The matrices M_1, M_2 and M_3 are all symmetrical if and only if one of the following conditions :

$(I)\ \alpha = q = \gamma = 0$, $(II)\ \alpha = p = \beta = 0$, $(III)\ \alpha = p = q = 0$ is satisfied.

We denote by $(K)_{123}$ the class of line complexes under investigation. Three subclasses of $(K)_{123}$ are examined : the line complexes $(K)^{I}_{123}$, $(K)^{II}_{123}$ and $(K)^{III}_{123}$ according to the conditions (I), (II) and (III) respectively.

2. The Subclass of Line Complexes $(K)^{I}_{123}$

Any line complex $K \in (K)^{I}_{123}$ will be characterized by the Pfaffian system of eqns. (3) with the condition (I) and the system of differential equations

$$\begin{bmatrix} dp \\ d\beta \\ dr \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \lambda_3 \\ \lambda_3 & 0 & \lambda_6 \\ \lambda_6 & 0 & \nu_6 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix} \quad \ldots(5)$$

in the third contact elements of the ray. From (3), (I) and (5), it follows that the number of independent parameters $N = 4(\lambda_1, \lambda_3, \lambda_6, \nu_6)$. Using Cartan's common method, we have the number of characteristic forms $\tilde{q} = 3(dp, d\beta, dr)$, the number of independent exterior from $S_1 = 2$, and the equality $\tilde{q} = S_1 + S_2$ gives that $S_2 = 1$. Hence, the Cartan's number $Q = S_1 + 2S_2 = 4 = N$. This means that, the Pfaffian system of eqns. (3) with the condition (I) is in involution and exists within one arbitary function of two arguments ($S_2 = 1$). Thus, we have the following existence theorem.

Theorem 1.1 — The range of existence of the line complexes $(K)^{I}_{123}$ comprises one arbitrary function of two arguments.

Therein, we investigate some geometrical properties of the line complexes $(K)^{I}_{123}$, which can help us to give their integral-free representation (i.e., geometrical construction) (Kovansov and Ponomarov 1975).

Property 1.1 — The centre $\bar{A}_4$ of the ray l is a double inflection centre (Kovansov 1963) and describes the coordinate plane $\bar{A}_4\bar{A}_1\bar{A}_2\{x^3 = 0\}$.

Property 1.2 — The complete integrable Pfaffian equation $\omega_3^2 = 0$, determines a hyperbolic holonomic line congruence C_I {two-dimensional submanifold of the Grassmann manifold $Gr(1, 3)$} belongs to a line complex $K \in (K)^{I}_{123}$. The focal surfaces of that line congruence degenerate into a plane curve {whose curvature equal to the invariant p} lying on the plane $x^3 = 0$ and the ideal line $\bar{A}_3\bar{A}_1$ which are described by the focal points $\bar{A}_4$ and $\bar{A}_3$ respectively.

Corollary 1.1 — Any line complex $K \in (K)^I_{123}$ admits a stratification into one-parameter families of the line congruences C_I (Kovansov 1963).

From the foregoing results, it follows that the geomertical construction of the line complexes $(K)^I_{123}$ is based on giving one-parameter family of plane curves {up to a function in two arguments}. Hence, we have the theorem which gives the integral-free representation of that subclass.

Theorem 1.2 — Take a one-parameter family of plane curves. For each curve, we construct a hyperbolic line congruence with degenerate focal surfaces consist of that curve and an ideal line passing through the invariant point. All these line congruences construct any line complex of the subclass $(K)^I_{123}$.

PROOF: The meaning of integral-free representation of a line complex is to obtain its Pfaffian system of equations from its geometrical construction. Therefore, let us take a one-parameter family of plane curves with layer consisting of the invariant ideal line ρ and a proper point. We take the improper vertices $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ such that they form the ideal plane with $\rho = \bar{A}_1\bar{A}_2$ {A_1 the invariant point} and $\bar{A}_4$ is the proper point. Henceforth, the layer is the plane $\bar{A}_4\bar{A}_2\bar{A}_1$ and the points $\bar{A}_4$, $\bar{A}_2$, $\bar{A}_1$, $\bar{A}_3$ are the vertices of a coordinate canonical tetrahedron conjugate to a line complex immersed in F_3 which generated by the ray $\bar{A}_4\bar{A}_3$. This line complex will be defined by the Pfaffian system of eqns. (3).

Since the plane $\bar{A}_4\bar{A}_2\bar{A}_1$ is fixed, then we have $d(\bar{A}_4\bar{A}_2\bar{A}_1) = 0$ and from (I) we obtain the Pfaffian equation $\omega_4^3 = 0$ $(*)$. Comparing the equation $(*)$ and the Pfaffian system of eqns. (3), we have $\alpha = q = \gamma = 0$ (I) which characterizes the constructed line complex of the subclass $(K)^I_{123}$.

3. THE SUBCLASS OF LINE COMPLEXES $(K)^{II}_{123}$

The line complexes $(K)^{II}_{123}$ are defined by the Pfaffian system of eqns. (3) under the condition (II) and the system of differential equations

$$\begin{bmatrix} dq \\ d\gamma \\ dr \end{bmatrix} = \begin{bmatrix} 0 & \mu_4 & \mu_5 \\ 0 & \mu_5 & \mu_6 \\ 0 & \mu_6 & \nu_6 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix} \qquad \ldots(6)$$

in the third order contact elements of the ray. By a similar way as that mentioned for the subclass $(K)^{I}_{123}$, it can be proved that the range of existence of the line complexes $(K)^{II}_{123}$ comprises one arbitrary function of two arguments.

We announce some results concerning the geometrical properties of $(K)^{II}_{123}$, especially their stratification.

Property 2.1 — If the ray $\bar{A}_4\bar{A}_3$ generates a line complex of $(K)^{II}_{123}$, then the ray $\bar{A}_4\bar{A}_2$ generates a parabolic line congruence C_{II} with focal surface degenerate into the ideal point $\bar{A}_2$.

Property 2.2 — When $\bar{A}_3$ {a double inflection centre} is fixed, the ray $\bar{A}_4\bar{A}_2$ will be fixed. This means that, the ray $\bar{A}_4\bar{A}_3$ generates a line bundle with vertex $\bar{A}_3$ and $\bar{A}_4\bar{A}_2$ as a layer.

Corollary 2.1 — Any line complex of the subclass $(K)^{II}_{123}$ can be stratified into two-parameter families of line bundles {one-dimensional submanifold of the Grassmann manifold $Gr(1, 3)$} with ideal vertices and the ray $\bar{A}_4\,\bar{A}_2$ as a layer.

From the preceding properties of this subclass, it is easy to see that one way of obtaining an integral-free representation of the line complexes $(K)^{II}_{123}$ is given by the following theorem.

Theorem 2.1 — Consider a parabolic line congruence C_{II} with focal surface degenerate into an ideal point. For each generator l of the line congruence C_{II}, construct a line bundle with an ideal vertex and layer l. All these line bundles represent a line complex of the subclass $(K)^{II}_{123}$.

3. The Subclass of Line Complexes $(K)^{III}_{123}$

In this section we shall consider the subclass $(K)^{III}_{123}$ of line complexes which are determined from the Pfaffian system (3) under the condition (III) and the system of differential equations

$$\begin{bmatrix} d\beta \\ d\gamma \\ dr \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_6 \\ 0 & 0 & \mu_6 \\ \lambda_6 & \mu_6 & \nu_6 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix} \quad \ldots(7)$$

in the third order contact elements of the ray.

Using Cartan's common method, it is easy to prove that the Pfaffian system of eqns. (3) with the condition (III) is in involution (has a solution). Its solution exists within three arbitrary functions of one variable $\{S_1 = 3\}$. Hence, we have the following existence theorem.

Theorem 3.1 — The range of existence of the line complexes $(K)_{123}^{III}$ comprises three arbitrary functions of a single argument.

Thereinafter we examine closely certain geometrical properties which can help us to give a representation without integrals of the subclass under discussion.

Remark : Every line complex of the subclass $(K)_{123}^{III}$ has the property that whose centres $\bar{A}_4$, $\bar{A}_3$ are simple inflection centres.

Property 3.1 — The complete integrable Pfaffian equation $\omega_3^2 = 0$, defines a hyperbolic holonomic linear line congruence C_{III} belonging to a line complex of the subclass $(K)_{123}^{III}$. The directrices of that line congruence are $\bar{A}_4\bar{A}_2$ and $\bar{A}_3\bar{A}_1$ which are described by the focal points $\bar{A}_4$ and $\bar{A}_3$ respectively.

Corollary 3.1 — Every line complex of the subclass $(K)_{123}^{III}$ admits a fibration into one-parameter families of the line congruences C_{III}.

Property 3.2 — For any line complex of the subclass $(K)_{123}^{III}$, the simple inflection centre $\bar{A}_4$ of the ray l describes a ruled surface with generator $l = \bar{A}_4\bar{A}_2$.

The above properties often make it is possible to intuit that, the geometrical construction of the line complexes $(K)_{123}^{III}$ is based on giving an arbitrary ruled surface {exists up to three functions of one argument} (Borovec̆ 1975).

Theorem 3.2 — Consider an arbitrary ruled surface Σ. For each generator l of Σ, we choose another ideal line L passing through the invariant point. The lines l and L are taken to be the directrices of a linear line congruence. The line complex $K \in (K)_{123}^{III}$ is obtained as a one-parameter family of such line congruences.

PROOF : We introduce a special family of frames $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$ where the vertices $\bar{A}_4$ and $\bar{A}_2$ are located on l, $\bar{A}_2$ is the intersection of l with the ideal plane and the vertices $\bar{A}_1$, $\bar{A}_3$ are placed on the ideal line L such that $\bar{A}_1\bar{A}_2$ is the invariant line with invariant point $\bar{A}_1$. We take l and L as the directrices of a linear line congruence generated by the ray $\bar{A}_4\bar{A}_3$.

Hence the points $\bar{A}_4$, $\bar{A}_3$, $\bar{A}_2$, $\bar{A}_1$ consist a canonical frame of a line complex, immersed in F_3, generated by the ray $\bar{A}_4\bar{A}_3$. This line complex is defined by the

Pfaffian system of eqns. (3). The ray $\bar{A}_4\bar{A}_2$ is a directrix of the linear line congruence C_{III} and generate the ruled surface Σ if and only if the differential

$$d(\bar{A}_4\bar{A}_2) = \omega_4^3(\bar{A}_3\bar{A}_2) + \omega_2^1(\bar{A}_4\bar{A}_1) + k\omega_3^2(\bar{A}_1\bar{A}_2)$$

is dependent only upon the differential form ω_3^2. Therefore, we obtain

$$\omega_4^3 \wedge \omega_3^2 = 0, \; \omega_2^1 \wedge \omega_3^2 = 0 \qquad \ldots(8)$$

Eqns. (3) and (8) give rise to the following condition

$$\alpha = p = q = 0 \qquad \text{(III)}$$

which characterizes every line complex that belongs to the subclass $(K)_{123}^{III}$.

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