Indian J. pure appl. Math., **13**(3) : 341–347, March 1982

A NOTE ON *FK*-SPACES CONTAINING Γ

B. CHOUDHARY, D. SOMASUNDARAM* AND S. BHATIA

Department of Mathematics, Indian Institute of Technology, New Delhi 110016

(*Received* 15 *April* 1980; *after revision* 3 *November* 1981)

The object of this note is to study inclusion theorems for the space Γ of entire functions defined over thc complex fields. By using functional analysis technique we unify some existing classical results.

Let Γ be the set of all sequences $x = \{x_n\}$ of complex numbers such that $|x_n|^{1/n} \to 0$ as $n \to \infty$. Let

$$\|x\| = \max \{ | x_0 | , \sup_{n \geqslant 1} | x_n |^{1/n} \}. \qquad \ldots(1)$$

It is known that Γ with this metric is non-normable complete linear metric space whose dual space is Γ^* where Γ^* is identified with the set of all sequences $y = \{y_n\}$ such that $| y_n |^{1/n}$ is bounded. For detailed discussion on Γ-spaces and their properties we refer Ganapathy Iyer (1948, 1950). Let e^n, $n = 0, 1, 2, \ldots,$ denotes the sequence $(0, 0, \ldots, 1, 0, 0, \ldots)$ with 'one' in the nth place and let e denotes the sequence $(1, 1, 1, \ldots)$.

The definition and basic properties of *FK*-spaces are given in Wilansky (1964) and Zeller (1951). Extending Zeller's Theorem [1951, Theorem 4.5 (a)] slightly, it follows from the closed graph theorem that if E and F are Frechet K-spaces with $E \subset F$, then E is continuously embedded in F. An *FK*-space whose topology is normable is called a *BK*-space. For preliminary ideas and useful results in this direction, we refer, in particular, Bennett (1973). We also refer Goffman and Pedrick (1965) where it is shown that Γ-space is an *FK*-space. The following sequence spaces will be important in our discussion:

c_0—the space of null sequences;

c—the space of all convergent sequences;

m—the space of all bounded sequences;

l—the space of all sequences $\{x_n\}$ such that

$$\sum_{n=0}^{\infty} | x_n | < \infty;$$

*Department of Mathematics, P.G. Centre, Madras University, Coimbatore 641041.

bv—the space of all sequences of bounded variation;

$bv_0 = bv \cap c_0$;

$bv = bv_0 + \{e\}$.

We denote by w the space of all real or complex valued sequences. We shall also be concerned with matrix transformations $y = Ax$, where $x, y \in w$, $A = \{a_{i,j}\}_{i,j=0}^{\infty}$ is an infinite matrix with complex coefficients.

We write

$$y_i = \sum_{j=0}^{\infty} a_{i,j} x_j \qquad (i = 0, 1, 2, \ldots).$$

If each of these series converges we say that $y = Ax$ exists and write $y \in w_A$. Let E be any subset of w. We denote by E_A the set

$$\{x \in w : Ax \text{ exists and } Ax \in E\}.$$

When $E = c$, c_A is called the convergence domain of the matrix A and each $x \in c_A$ is said to be A-limitable.

We will need the following theorem of Zeller (1951).

Theorem A — Let E be an FK-space whose topology is given by the semi-norms $\{q_n\}_{n=0}^{\infty}$ and let A be an infinite matrix. Then E_A is an FK-space when topologized by

$$x \to | x_j | \; (j = 0, 1, 2, \ldots);$$

$$x \to \sup_n \left| \sum_{j=0}^{n} a_{i,j} x_j \right| \qquad (i = 0, 1, 2, \ldots);$$

$$x \to q_n(Ax) \; (n = 0, 1, 2, \ldots).$$

We now remark that the topology of Γ given by (1) is the same as that defined by a suitable system of semi-norms (as in Zeller 1951). Let $\{R_i\}$ be any increasing sequence of positive numbers with $R_i \to \infty$ as $i \to \infty$. Define, for $x = \{x_n\}$, $x \in \Gamma$,

$$p_i(x) = \sup_{n \geqslant 0} | x_n R_i^n |. \qquad \ldots(2)$$

This is, in fact, a norm, it is certainly defined when $x \in \Gamma$. Let $\{x^j\}$, $x^j = \{x_n^j\}$ be a sequence of elements of Γ. We have to show that if $x^j \to 0$ in accordance with Zeller's definition, then

$$\|x^j\| \to 0 \text{ [with our definition (1)]}$$

and conversely.

Suppose first that $\|x^j\| \to 0$. We have to show that, for every fixed i,

$$p_i(x^j) \to 0 \qquad \ldots(3)$$

as $j \to \infty$, so take a fixed i. Take $\epsilon \leqslant 1/R_i$. We have, for sufficiently large j,

$$\|x^j\| < \epsilon;$$

that is, by definition

$$|x_0^j| < \epsilon;$$

$$|x_n^j| < \epsilon^n \qquad (n \geqslant 1).$$

Then

$$|x_n^j R_i^n| \leqslant \begin{cases} \epsilon \text{ if } n = 0 \\ \epsilon^n R_i^n \leqslant \epsilon R_i \end{cases} \qquad \left(\text{since } \epsilon < \frac{1}{R_i}\right) \text{ if } n \geqslant 1.$$

Thus

$p_i(x^j) < \max(\epsilon, \epsilon R_i)$, and this can be made arbitrarily small by choice of ϵ.

Conversely, suppose that (3) holds for every fixed i. Given any $\epsilon > 0$, choose i so that $R_i > 1/\epsilon$. Once we have fixed i, we have, for all sufficiently large j,

$$p_i(x^j) < \epsilon.$$

That is to say, for all n

$$|x_n^j| < \epsilon/R_i^n < \epsilon^{n+1}.$$

Thus, supposing as we may that $\epsilon < 1$ we have

$$|x_0^j| < \epsilon;$$

$$|x_n^j| < q^{((n+1)/n)} < \epsilon \ (n \geqslant 1),$$

so that

$$\|x^j\| \leqslant \epsilon.$$

We remark that Theorem 1 of Ganapathy Iyer (1950), though stated in a different form, is equivalent to an assertion similar to the result just proved, but with the semi-norms

$$q_i(x) = \sum_{n=0}^{\infty} |x_n| R_i^n.$$

Following Zeller, we consider an *FK*-space E with its topology defined by a set of semi-norms $\{p_i(x)\}$. Let X be any set of elements of E. It would appear natural to define X as bounded if

(A) For each i, $p_i(x)$ is bounded for $x \in X$ (the bound possibly depending on i).

In the special case in which $E = \Gamma$, it would also appear natural to define X as bounded if

(B) With $\|x\|$ defined by (1), $\|x\|$ is bounded for $x \in X$.

However, with the set of semi-norms for Γ just introduced, we shall see that the special case of (A) in which E is Γ does not agree with (B). Accordingly, we shall describe a subset X of any FK-space as bounded if (A) holds, and we shall describe a subset X of Γ as quasi-bounded if (B) holds.

It is easy to see that any bounded subset of Γ is quasi-bounded. But the converse is false. For, suppose we consider the set of elements x^j defined by

$$x^j = \{x_n^j\}$$

where

$$x_n^j = \begin{cases} 1 & (n \leqslant j) \\ 0 & (n > j). \end{cases}$$

Clearly, for each j, $x^j \in \Gamma$, also $\|x^j\| = 1$. So the set is quasi-bounded. But if $R_i > 1$ (which must happen for sufficiently large i, since $R_i \to \infty$ as $i \to \infty$) we have

$$p_i(x^j) = R_i^j$$

which, for fixed i, is not bounded.

We now prove a theorem under the assumption that E is a BK-space.

Theorem 1 — Let E be an FK-space, the topology being defined by the set of semi-norms $q_0(x)$, $q_1(x)$, ... In order that $\Gamma \subset E$ it is necessary that $e^n \in E$ (all n) and that, for each i, there is an $R(i)$ such that

$$q_i(e^n/(R(i))^n) \qquad \ldots(4)$$

is bounded for all n.

It is sufficient that this should hold with $R(i)$ having the same value for every i.

Note that the assertion that (e^n/R^n) is bounded means that (4) holds with $R(i) = R$. In the special case in which E is a BK-space the topology is defined by just one norm, so that the necessary and sufficient conditions are the same.

Proof: To prove necessity, it is trivially necessary that $e^n \in E$. Now consider any fixed i. If $\Gamma \subset E$ then $q_i(x)$ restricted to Γ, is a semi-norm on Γ. Hence, by Satz 3.2 of Zeller (1951) there is an integer r and a constant M such that, for $x \in \Gamma$,

$$q_i(x) \leqslant M(p_0(x) + p_1(x) + \ldots + p_r(x)).$$

(In general, r will depend on i). But, for the semi-norms given by (2) we have

$$p_i(x) \leqslant p_j(x) \text{ for } i < j$$

(this would not apply to *FK*-spaces in general). It therefore follows that

$$q_i(x) = 0\ (p_r(x)).$$

If we take $R = R_r$ then $p_r(e^n/R^n)$ is bounded for all n, hence so is $q_n(e^n/R^n)$.

Conversely, suppose that $\{e^n/R^n\}$ is a bounded subset of E. Fix any element x of Γ. Trivially, we can write

$$x = \sum_{k=0}^{\infty} x_k e^k/R^k$$

where

$$\sum_{k=0}^{\infty} |\ x_k\ | < \infty.$$

If the topology of E is given by the set of semi-norms $\{p_i\}$ then, for any i,

$$p_i\Big(\sum_{j=m}^{n} x_j e^j/R^j\Big) \leqslant \sum_{j=m}^{n} |\ x_j\ | \sup_j p_i(e^j/R^j) \to 0 \text{ as } m,\ n \to \infty.$$

Thus

$$\Big\{\sum_{j=0}^{n} x_j e^j/R^j\Big\}_{n=0}^{\infty} \text{ is a Cauchy sequence in } E.$$

Thus, since E is complete, it converges in E. Thus $x \in E$ and, since x is any element of Γ, it follows that Γ is contained in E.

Remark : It is of some slight interest that, even if we restrict E to be a *BK*-space, it is still not necessarily true that if Γ is contained in E and if X is quasi-bounded in Γ then X is bounded in E. Take $R = 1$. Let E be the space of sequences $x = \{x_n\}$ with $(n + 1)\, x_n$ bounded, normed by

$$\|x\| = \sup_n |\ (n + 1)\, x_n\ |\ .$$

This clearly, is not only an *FK*-space but is, in fact, a *BK*-space. It clearly includes Γ; but $\{e^n\}$ is not bounded in E.

We have another simple counter example. If we take $R < 1$ and $E = m$. Then clearly, this includes Γ; but $\{e^n/R^n\}$ is not bounded in m.

Combining Theorem A and Theorem 1 we now prove a result which we call Corollary 1.

Let column 'k' be denoted by $C^{(k)}$ so that $C^{(k)} = \{a_{i,k}\}$. We will prove corollary 1 under the assumption that E is a *BK*-space.

Corollary 1 — Let A be a matrix and E an *FK*-space. Then A maps Γ into E if and only if there is some $R > 0$ such that

$$\{C^{(k)}/R^{(k)}\}$$

is a bounded subset of elements of E.

PROOF : The topology of E is now given by a single norm which we denote by $\| \|_E$. Then, by Theorem A, E_A is an FK-space whose topology is given by the set of semi-norms:

(i) $x \to | x_j | \; (j = 0, 1, 2, \ldots)$

(ii) $x \to \sup_n | \sum_{j=0}^{n} a_{i,j} x_j | \quad (i = 0, 1, 2, \ldots)$

(iii) $\|Ax\|_E$.

The necessity is given by taking the necessity part of Theorem 1 applied to the particular semi-norm $\|Ax\|_E$.

We will denote the above expression (i) by $q_j(x)$ and (ii) by $r_j(x)$.

For sufficiency, we are given that, for some R

$$\|C^{(k)}/R^k\|_E$$

is bounded, or, what is the same thing, that

$$\|Ae^k/R^k\|_E \text{ is bounded.}$$

Note that we use the particular nature of the semi-norms to deduce that e^k/R^k is bounded in E_A. We are given that the last semi-norm is bounded, so we have to show that, for fixed j,

$$q_j(e^k/R^k) \text{ is bounded} \tag{I}$$

$$r_j(e^k/R^k) \text{ is bounded.} \tag{II}$$

By definition

$$q_j(e^k/R^k) = \begin{cases} 1/R^j & (k = j) \\ 0 & (k \neq j). \end{cases}$$

For fixed j, this is bounded whatever R.

Also

$$r_j(e^k/R^k) = | a_{i,k}/R^k | .$$

But $\frac{a_{i,k}}{R^k}$ is coordinate i of $C^{(k)}/R^{(k)}$; so, since the coordinate functionals in E are continuous, (II) follows from the boundedness of

$$\|C^{(k)}/R^{(k)}\|.$$

We now apply the sufficiency part of Theorem 1. Hence the Corollary 1 is proved.

We note that many results can be obtained from Corollary 1 by substituting different spaces for E. As illustrations we give:

Corollary 2 — A matrix A maps Γ into c if and only if there is some $R > 0$ such that

(i) $\lim_{i \to \infty} a_{i,j} = a_j$ exists for each j;

(ii) $\sup_{i,j} \frac{|a_{i,j}|}{R^j} < \infty$.

Corollary 3 — A martix A maps Γ into l if and only if there is some $R > 0$ such that

(i) $\sum_{i=0}^{\infty} |a_{i,j}| < \infty$ for each fixed j;

(ii) $\sup_j \sum_{i=0}^{\infty} \frac{|a_{i,j}|}{R^j} < \infty$.

Corollary 4 — A matrix A maps Γ into bv if and only if there is some $R > 0$ such that

(i) $\sum_{i=0}^{\infty} |a_{i+1,j} - a_{i,j}| < \infty$ for each fixed j;

(ii) $\sup_j \frac{1}{R^j} \left\{ |a_{0,j}| + \sum_{i=0}^{\infty} |a_{i+1,j} - a_{i,j}| \right\} < \infty$.

Remark : We cannot apply the technique of this paper to get inclusion theorems from Γ^* to other well known sequence spaces, since Γ^* is not even a linear metric space (see Ganapathyl Iyer 1948).

Acknowledgement

The authors express their sincere gratitude to Professor B. Kuttner whose generous help led to the improvement of this paper. A preliminary report concerning this note was published in the Notices of American Mathematical Society in the year 1974.

References

Bennett, G. (1973). Some inclusion theorems for sequence spaces. *Pacific J. Math.*, **46**, 17-30.

Chandrasekhara, Rao, K. (1969). Matrix transformations of some sequence spaces. *Pacific J. Math.*, **31**, 171-74.

——————(1970). Matrix transformations of some sequence space II. *Glasgow Math. J.*, **11**, 162–66.

Ganapathy, Iyer, V. (1948). On the space of Integral Functions I. *J. Indian math. Soc.*, **12**, 13–30.

——————(1950). On the space of Integral Functions II. *Quart. J. Math.*, **1**, 86–96.

Goffman, C., and Pedrick, G. (1965). First Course in Functional Analysis, Prentice-Hall, New Jersey.

Jacob, Roy, T. (Jr) (1977). Matrix transformations involving simple sequence spaces. *Pacific J. Math.*, **70**, 179–87.

Wilansky, A. (1964). Functional Analysis. Blaisdell, New York.

Zeller, K. (1951). Allgemeine Eigenschaften von Limitierunge-Verfahren. *Math. Z.*, **53**, 463–87.