Indian J. pure appl. Math., 13(3) : 348–361, March 1982

CERTAIN PROPERTIES OF A DISTRIBUTIONAL GENERALIZED WHITTAKER TRANSFORM

A. K. TIWARI AND AYE KO

Department of Mathematics, Regional College of Engineering, Nagpur 440011

(*Received* 17 *July* 1980; *after revision* 2 *June* 1981)

In this paper an extension of the generalized Whittaker transform

$$F(x) = \int_0^\infty (xt)^{\sigma-(1/2)} \exp\left(-\tfrac{1}{2} qxt\right) W_{k,m}(pxt) f(t)\, dt$$

to generalized functions (distributions) is provided. Some Abelian theorems for the distributional generalized Whittaker transform are proved. The validity of the inversion formula in the distributional sense is established.

1. INTRODUCTION

Some well-known generalizations of the classical Laplace transform

$$F(x) = \int_0^\infty e^{-xt} f(t)\, dt \qquad \ldots(1.1)$$

include the generalized Whittaker transform

$$F(x) = \int_0^\infty (xt)^{\sigma-(1/2)} \exp\left(-\tfrac{1}{2} qxt\right) W_{k,m}(pxt) f(t)\, dt \qquad \ldots(1.2)$$

due to Srivastava (1968). For $\sigma = m$, $k + m = \frac{1}{2}$, $p = q = 1$, (1.2) reduces to (1.1). Srivastava (1968) proved in the classical sense the following complex inversion formula (under certain conditions) for the above generalization (1.2):

Theorem (Srivastava 1968, pp. 387–88) — If $F(x)$ is given by (1.2), then

$$f(t) = \int_0^\infty \Phi(xt) F(x)\, dx \qquad \ldots(1.3)$$

where

$$\Phi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\Psi(1-s)}\, ds$$

and

$$\Psi(s) = \frac{p^{m+(1/2)}\,\Gamma(\sigma \pm m + s)}{[\frac{1}{2}(q+p)]^{\sigma+m+s}\,\Gamma(\sigma - k + s + \frac{1}{2})}$$

$$\times\ {}_2F_1\left[\begin{matrix}\sigma + m + s, m - k + \frac{1}{2}\,; \\ \\ \sigma - k + s + \frac{1}{2}\,;\end{matrix}\quad \frac{q-p}{q+p}\right]$$

provided $|\Phi(x)|$ exists, (1.3) is convergent

$$x^{-\sigma} F(x) \in L(0, \infty),\ t^{c-1} f(t) \in L(0, R_0),\ R_0 > 0,$$

and Re $(\sigma \pm m + 1) > c > 0$. {See also Srivastava and Vyas (1969)}

Zemanian (1965) discussed some classical properties of the Laplace transform (1.1) for ordinary functions and then developed in detail the same properties for distributions. Similar work is done by Rao (1978) for distributional ${}_1F_1$-transform. We propose to discuss here Srivastava's generalized Whittaker transform (1.2) for ordinary functions and then extend the same for distributions. The complex inversion formula (1.3) will also be extended to distributions.

The space D of testing functions consists of all complex valued functions $\phi(t)$ that are infinitely smooth and zero outside some finite interval. S denotes the space of testing functions of rapid descent. The spaces D' and S' are the dual spaces of the spaces D and S respectively. Distributions whose supports are bounded on the left are called right sided distributions and are denoted by D'_R. A distribution is said to be left sided if its support is bounded on the right. D'_L represents the space of such distributions.

A function $\phi(t)$ defined on $0 < t < \infty$ is said to be a member of $M_{c,d,\alpha}$ if $\phi(t)$ is infinitely differentiable and

$$\begin{aligned} i_{c,d,l}(\phi) &= \sup_{0<t<\infty} |\lambda_{c,d}(t)\,(-\,tD_t)^l\,\{t\phi(t)\}| \\ &\leqslant C_l A^l l^{l\alpha} \end{aligned} \qquad \ldots(1.4)$$

where

$$\lambda_{c,d}(t) = \begin{cases} t^{-c}, & 0 < t \leqslant 1 \\ t^{-d}, & 1 < t < \infty, \end{cases} \qquad \ldots(1.5)$$

$\alpha \geqslant 0,\ l = 0, 1, 2, \ldots.$

The constants $A_{l\alpha}$ and C_l depend on the testing function ϕ. For $l = 0$ we set $l^{l/\alpha} = 1$. We define the countable union space $M_{c,d}$ by

$$M_{c,d} = \bigcup_{\alpha_i=1}^{\infty} M_{c,d,\alpha\,i}. \qquad \ldots(1.6)$$

Thus $M_{c,d}$ is the space of all testing functions $\phi(t)$ on $0 < t < \infty$ such that

$$i_{c,d,l}(\phi) = \sup_{0<t<\infty} | \lambda_{c,d}(t) (-tD_t)^l \{t\phi(t)\} |$$

$$< \infty.$$

It can be easily proved that $D(I) \subset M_{c,d,\alpha}$. $D(I)$ is space of all smooth functions on I $(0 < t < \infty)$ having compact support. The spaces $M'_{c,d,\alpha}$ and $M'_{c,d}$ denote the dual of the spaces $M_{c,d,\alpha}$ and $M_{c,d}$ respectively.

2. The Generalized Whittaker Transform of Ordinary Functions

Let $f(t)$ be a locally integrable function satisfying the following conditions:

Conditions A

(i) $f(t) = 0$ for $-\infty < t < T$.

(ii) There exists a real number C such that $e^{-ct} f(t)$ is absolutely integrable over $-\infty < t < \infty$. The generalized Whittaker transform of $f(t)$ is defined by

$$L[f(t)] = F(s) = \int_0^\infty f(t)\, \theta(s, t)\, dt \qquad \ldots(2.1)$$

where

$$\theta(s, t) = (st)^{\sigma-(1/2)} \exp(-\tfrac{1}{2} qst)\, W_{k,m}(pst).$$

Since $f(t) = 0$ for $-\infty < t < T$, we write (2.1) as

$$F(s) = \int_T^\infty f(t)\, \theta(s, t)\, dt. \qquad \ldots(2.2)$$

We shall refere to (2.2) as a right sided generalized Whittaker transform. If $T < 0$, (2.2) can be written as

$$F(s) = \int_T^0 f(t)\, \theta(s, t)\, dt + \int_0^\infty f(t)\, \theta(s, t)\, dt$$

$$= I_1 + I_2.$$

We write

$$I_2 = \int_0^\infty \left[\exp\left\{-\left(\frac{qs - 2C}{2}\right)t\right\} (st)^{\sigma-(1/2)}\; W_{k,m}(pst)\right] [e^{-ct} f(t)]\, dt.$$

Denoting

$$\exp\left\{-\left(\frac{qs - 2C}{2}\right)t\right\} (st)^{\sigma-(1/2)}\, W_{k,m}(pst) \text{ by } K(s,t)$$

and using the order properties of $W_{k,m}(z)$ (Whittaker and Watson 1946) viz.

$$\begin{aligned} W_{k,m}(z) &= 0\,(z^k e^{-(z/2)}),\ |z| \to \infty \\ &= 0\,(z^{(1/2)+m}),\ |z| \to 0,\ m < 0 \\ &= 0\,(z^{(1/2)-m}),\ |z| \to 0,\ m > 0. \end{aligned}$$

We have for $t \to \infty$

$$|K(s,t)| \leqslant M \left| \exp\left\{ -\left(\frac{qs - 2C + ps}{2}\right) t \right\} (st)^{\sigma-(1/2)+k} \right|$$

$$|K(s,t)| \to 0 \text{ as } t \to \infty \text{ for } \operatorname{Re}\{(q+p)s\} > 2C.$$

Remark : For $p = q = 1$, we get the result obtained in (Zemanian 1965, p. 214).

Considering the order property of $W_{k,m}(z)$ for small t and $m > 0$, we have

$$|K(st)| \leqslant M \left| \exp\left\{ -\left(\frac{qs - 2C}{2}\right) t \right\} (st)^{\sigma-m} (p)^{(1/2)-m} \right|$$

(M is some constant)

which is finite at $t = 0$ if $\operatorname{Re}(\sigma - m) \geqslant 0$.

Thus for $\operatorname{Re}\{(q+p)s\} > 2C$, $m > 0$, $\operatorname{Re}(\sigma - m) \geqslant 0$ the integrals $\int_T^\infty \theta(s,t)\, f(t)\; dt$ converges absolutely in the half plane $\operatorname{Re}\{(q+p)s\} > 2C$.

3. The Generalized Whittaker Transform of Right Sided Distributions

We define the generalized Whittaker transform of a distribution $f(t)$ whose support is bounded on the left by

$$\begin{aligned} F(s) &= L\{f(t)\} \\ &= \langle f(t), \theta(s,t) \rangle. \end{aligned} \qquad \ldots(3.1)$$

Assuming that there exists a real number C for which $e^{-ct} f(t) \in S'$, (3.1) can be written as

$$F(s) = \langle e^{-ct} f(t), \lambda(t)\, \theta(s,t)\, e^{ct} \rangle \qquad \ldots(3.2)$$

where $\lambda(t)$ is a smooth function with support bounded on the left which equals one over a neighbourhood of the support of $f(t)$. It is easy to prove that

$$\lambda(t)\, \theta(s,t)\, e^{ct} \in S$$

for $\qquad \operatorname{Re}\{(q+p)s\} > 2C, \qquad \operatorname{Re}(\sigma - m) \geqslant 0,\ m > 0.$

Thus the right-hand side of (3.2) possesses a sense as the application of a distribution in S' to a testing function in S.

4. Abelian Theorem for the Generalized Whittaker Transform of Ordinary Functions

Theorem 4.1 — If the locally integrable function $f(t)$ satisfies conditions A of the section 2 with $T = 0$ and if there exist a complex number α and a real number $\eta(\eta > -1)$ such that

$$\lim_{t\to 0+} \frac{f(t)}{t^{\eta}} = \alpha \qquad \ldots(4.1)$$

then

$$\lim_{x\to\infty} \frac{x^{\eta+1}F(x)}{B(\sigma, \eta, m, k, q, p)} = \alpha \qquad \ldots(4.2)$$

where

$$B(\sigma, \eta, m, k, q, p) = \frac{p^{m+(1/2)}\,\Gamma(\sigma \pm m + \eta + 1)}{[\frac{1}{2}(q+p)]^{\sigma+m+\eta+1}\,\Gamma(\sigma - k + \eta + \frac{3}{2})} \times {}_2F_1\left[\begin{matrix}\sigma + m + \eta + 1, m - k + \frac{1}{2}\,; \\ \sigma - k + \eta + \frac{3}{2}\,;\end{matrix}\ \frac{q-p}{q+p}\right] \qquad \ldots(4.3)$$

and

$$F(x) = \int_0^\infty \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)}\,W_{k,m}(pxt)\,f(t)\,dt. \qquad \ldots(4.4)$$

Proof : For $\eta > -1$ and $x > 0$ we have

$$\int_0^\infty t^{\eta}\exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)}\,W_{k,m}(pxt)\,dt$$

$$= \frac{1}{x^{\eta+1}}\int_0^\infty \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma+\eta-(1/2)}\,W_{k,m}(pxt)\,d(xt). \qquad \ldots(4.5)$$

By using the result (Srivastava 1968, p. 388) viz.

$$\int_0^\infty u^{\sigma-\xi-(1/2)}\,e^{-(1/2)qu}\,W_{k,m}(pu)\,du = \Psi(1-\xi)$$

where

$$\Psi(\xi) = \frac{p^{m+(1/2)}\,\Gamma(\sigma \pm m + \xi)}{[\frac{1}{2}(q+p)]^{\sigma+m+\xi}\,\Gamma(\sigma - k + \xi + \frac{1}{2})}$$

(*equation continued on p.* 353)

$$\times {}_2F_1\left[\begin{matrix}\sigma + m + \xi, m - k + \frac{1}{2};\\ \sigma - k + \xi + \frac{1}{2};\end{matrix}\ \frac{q-p}{q+p}\right].$$

We can write

$$\int_0^\infty t^\eta \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\,dt$$

$$= \frac{1}{x^{\eta+1}} B(\sigma, \eta, m, k, q, p) \qquad \ldots(4.6)$$

where

$$B(\sigma, \eta, m, k, q, p) = \frac{p^{m+(1/2)}\,\Gamma(\sigma \pm m + \eta + 1)}{[\frac{1}{2}(q+p)]^{\sigma+m+\eta+1}\,\Gamma(\sigma - k + \eta + \frac{3}{2})}$$

$$\times {}_2F_1\left[\begin{matrix}\sigma + m + \eta + 1, m - k + \frac{1}{2}\\ \sigma - k + \eta + \frac{3}{2}\end{matrix}\ \frac{q-p}{q+p}\right].$$

Note : For $\sigma = m, k + m = \frac{1}{2}, p = q = 1$ we have

$$B(\sigma, \eta, m, k, q, p) = \Gamma(\eta + 1)$$

and the result (4.6) reduces to the result of (Zemanian 1965, p. 243).

By using (4.6) and assuming that $x > 0$ and $y > 0$ we may write,

$$\left| x^{\eta+1}F(x) - \alpha B(\sigma, \eta, m, k, q, p)\right|$$

$$= \left| x^{\eta+1}\int_0^\infty \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\{f(t) - \alpha t^\eta\}\,dt\right|$$

$$\leqslant x^{\eta+1}\left|\int_0^Y \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\{f(t) - \alpha t^\eta\}\,dt\right|$$

$$+ x^{\eta+1}\left|\int_Y^\infty \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\{f(t) - \alpha t^\eta\}\,dt\right|$$

$$= I_1 + I_2. \qquad \ldots(4.7)$$

Now, I_1

$$= \left| x^{\eta+1}\int_0^Y t^\eta \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\left\{\frac{f(t)}{t^\eta} - \alpha\right\}dt\right|$$

$$\leqslant x^\eta \sup_{0<t<y}\left|\frac{f(t)}{t^\eta} - \alpha\right| \int_0^Y t^\eta \exp\left(-\tfrac{1}{2}qxt\right)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\,d(xt)$$

(*equation continued on p.* 354)

$$\leqslant \sup_{0<t<y} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| x^{\eta} \int_{0}^{\infty} t^{\eta} \exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\, d(xt)$$

$$\leqslant \sup_{0<t<y} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| B(\sigma, \eta, m, k, q, p). \qquad \ldots(4.8)$$

For I_2, according to the second of conditions A and the assumption that $C > 0$,

$$e^{-ct} \mid f(t) - \alpha t^{\eta} \mid \text{ is absolutely integrable over } 0 < t < \infty.$$

Hence

$$I_2 = x^{\eta+1} \left| \int_{Y}^{\infty} \exp(-\tfrac{1}{2} qxt)\, (xt)^{\sigma-(1/2)}\, W_{k,m}(pxt)\, e^{ct} e^{-ct} \{f(t) - \alpha t^{\eta}\}\, dt \right|$$

$$\leqslant x^{\eta+1} \left| \int_{Y}^{\infty} \exp\left\{-\left(\frac{qx - 2c}{2}\right)t\right\} (xt)^{\sigma-(1/2)}\, W_{k,m}(pxt) \times e^{-ct}\{f(t) - \alpha t^{\eta}\}\, dt \right|.$$

Now the function $\exp\left\{-\left(\frac{qx - 2c}{2}\right) t\right\} (xt)^{\sigma-(1/2)} W_{k,m}(pxt)$ is finite and continuous in $y < t < \infty$ and tends to zero as $t \to \infty$ for Re $(q + p)\, x > 2C$. Let the upper bound of the function be attained at the point $t = \beta$, then

$$I_2 \leqslant x^{\eta+1} \left| \exp\left\{-\left(\frac{qx - 2c}{2}\right)\beta\right\} (x\beta)^{\sigma-(1/2)} W_{k,m}(px\beta) \right| \times \int_{Y}^{\infty} e^{-ct} \mid f(t) - \alpha t^{\eta} \mid dt$$

$$= M x^{\eta+1} \exp\left\{-\left(\frac{qx - 2c}{2}\right)\beta\right\} (x\beta)^{\sigma-(1/2)} W_{k,m}(px\beta) \qquad \ldots(4.9)$$

where $\quad M = \int_{0}^{\infty} e^{-ct} \mid f(t) - \alpha t^{\eta} \mid dt.$

Now let $\epsilon > 0$. Because of (4.1) we can choose Y so small that the right-hand side of (4.8) which is independent of x, becomes less than ϵ. By fixing y in this way, we can choose x so small that the right-hand side of (4.9) also becomes less than ϵ. Since ϵ is arbitrary, the left-hand side of (4.7) can be made arbitrary small for all sufficiently large x which in turns proves (4.2).

Now we shall extend Theorem 4.1 to distributions:

Theorem 4.2 — Let $f(t)$ be a generalized Whittaker transformable distribution having its support in $0 < t < \infty$, and assume that over some neighbourhood of the origin $f(t)$ is a regular distribution corresponding to a (Lebesgue) integrable function $h(t)$. Also assume that there exist a complex number α and a real number $\eta > -1$ such that

$$\lim_{t\to 0+} \frac{h(t)}{t^{\eta}} = \alpha t. \qquad \ldots(4.10)$$

Then

$$\lim_{x\to\infty} \frac{x^{\eta+1}F(x)}{B(\sigma, \eta, m, k, q, p)} = \alpha. \qquad \ldots(4.11)$$

PROOF : The proof is similar to the proof of Theorem 8.6 – 2 of Zemanian (1965, p. 247).

5. A FINAL VALUE THEOREM FOR THE GENERALIZED WHITTAKER TRANSFORM

In this section we shall relate the behaviour of $f(t)$ as $t \to \infty$ to the behaviour of $F(x)$ as $x \to 0+$.

Theorem 5.1 — If the locally integrable function $f(t)$ satisfies conditions A of section 2 and if there exist a complex number α and a real number $\eta(\eta > -1)$ such that

$$\lim_{t\to\infty} \frac{f(t)}{t^{\eta}} = \alpha \qquad \ldots(5.1)$$

then

$$\lim_{x\to 0+} \frac{x^{\eta+1}F(x)}{B(\sigma, \eta, m, k, q, p)} = \alpha \qquad \ldots(5.2)$$

where $B(\sigma, \eta, m, k, q, p)$ and $F(x)$ are as given in Theorem 4.1.

PROOF : Let us assume that the support of $f(t)$ is bounded on the left of $t = T, T \geqslant 0$.

As in Theorem 4.1 we can prove

$$\int_0^\infty t^{\eta} \exp\{-\tfrac{1}{2} qxt\} (xt)^{\sigma-(1/2)} W_{k,m}(pxt)\, dt$$

$$= \frac{1}{x^{\eta+1}} B(\sigma, \eta, m, k, q, p).$$

Also as in Theorem 4.1 we write

$$| x^{\eta+1}F(x) - \alpha B(\sigma, \eta, m, k, q, p) |$$

$$\leqslant | x^{\eta+1} \int_0^y \exp\{-\tfrac{1}{2} qxt\} (xt)^{\sigma-(1/2)} W_{k,m}(pxt) \{f(t) - \alpha t^{\eta}\}\, dt |$$

(*equation continued on p* 356)

$$+ \left| x^{\eta+1} \int_y^\infty \exp \{(-\tfrac{1}{2} qxt)\} (xt)^{\sigma-(1/2)} W_{k,m}(pxt) \{f(t) - \alpha t^\eta\} dt \right|$$

$$= I_1 + I_2. \qquad \ldots(5.3)$$

Let us first consider I_2,

$$I_2 \leqslant x^{\eta+1} \int_Y^\infty t^\eta \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt) \left| \left\{ \frac{f(t)}{t^\eta} - \alpha \right\} \right| dt$$

$$\leqslant \sup_{\leqslant y \leqslant t < \infty} \left| \frac{f(t)}{t^\eta} - \alpha \right| \int_0^\infty \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(xt)\, d(xt)$$

$$= B(\sigma, \eta, m, k, q, p) \sup_{y \leqslant t < \infty} \left| \frac{f(t)}{t^\eta} - \alpha \right|. \qquad \ldots(5.4)$$

Let ϵ be a fixed arbitrary positive number. From (5.1) it is clear that we can choose y so large that the right-hand side of (5.4) which is independent of x can be made less than ϵ. Now for I_1

$$I_1 = x^{\eta+1} \left| \int_0^Y \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt) [f(t) - \alpha t^\eta] dt \right|$$

having fixed y we can choose x so small that I_1 becomes less than ϵ. Thus from (5.3)

$$| x^{\eta+1} F(x) - \alpha B(\sigma, \eta, m, k, q, p) | \leqslant \epsilon.$$

This proves (5.2) for $T \geqslant 0$. If $T < 0$ then the additional term

$$I_3 = \left| x^{\eta+1} \int_T^0 \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt) f(t) dt \right.$$

occurs in the right-hand side of (5.3).

The function

$$\theta(x, t) = \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt)$$

is continuous and finite within the interval $(T, 0)$, hence it must be bounded. Let the upper bound of $\theta(x, t)$ be attained at $t = \beta$, then

$$I_3 \leqslant \left| x^{\eta+1} \exp(-\tfrac{1}{2} qx\beta) (x\beta)^{\sigma-(1/2)} W_{k,m}(px\beta) \int_T^0 f(t) dt \right|$$

as $x \to 0+$, $I_3 \to 0$.

This completes the proof of Theorem 5.1.

Now we extend the Theorem 5.1 to distributions.

Theorem 5.2 — Let $f(t)$ be a generalized Whittaker transformable distribution in D'_R. Assume that $f(t)$ is a regular distribution corresponding to a locally integrable function $h(t)$ that satisfies the second of condition A, over some semi-infinite interval $T < t < \infty$. Also assume that there exist a complex number α and a real number $\eta(\eta > -1)$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t^{\eta}} = \alpha \qquad \ldots(5.5)$$

then

$$\lim_{x\to 0+} \frac{x^{\eta+1}F(x)}{B(\sigma, \eta, m, k, q, p)} = \alpha. \qquad \ldots(5.6)$$

PROOF : The proof is similar to the proof of Theorem 8.7 – 2 of (Zemanian 1965, p. 250).

6. COMPLEX INVERSION THEOREM

Theorem 6.1 — For any real $x > 0$, $\exp(-\frac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt)$ is a member of $M_{c,d}$ for $c < \sigma - m + 1$ and every real number d.

PROOF : For $\exp(-\frac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt)$ to be in $M_{c,d}$ we have to show that

$$\sup_{0<t<\infty} |\lambda_{c,d}(t) (-tD_t)^K \{t\psi(t)\}| \text{ is finite where}$$

$$\psi(t) = \exp(-\tfrac{1}{2} qxt) (xt)^{\sigma-(1/2)} W_{k,m}(pxt).$$

We have

$$\sup_{0<t<\infty} |\lambda_{c,d}(t) (-tD_t)^K \{t\psi(t)\}|$$

$$= \sup_{0<t<\infty} |\lambda_{c,d}(t) (-1)^K \Sigma C_r t^{r+1} D_t^r \psi(t)|,$$

$$0 \leqslant r \leqslant K. \qquad \ldots(6.1)$$

Where C_r's are some constants.

Considering (6.1) for $1 < t < \infty$ we have.

$$\sup_{1<t<\infty} |t^{-d}(-1)^K \sum_r C_r t^{r+1} D_t^r \{\exp(-\tfrac{1}{2} qxt) (xt)^{-(1/2)} W_{k,m}(pxt)\}|. \qquad \ldots(6.2)$$

Using the order properties of $W_{k,m}(z)$ (Whittaker and Watson 1946) it can be easily seen that as $t \to \infty$, (6.2) is bounded for any value of d provided Re $(q + p) > 0$.

Considering (6.1) for $0 < t < 1$ we have

$$\sup_{0<t<1} \left| t^{-c}(-1)^{K} \sum_{r} C_r t^{r+1} D_t^r \{\exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt)\} \right|. \qquad ...(6.3)$$

Again using the order properties of $W_{k,m}(z)$ as $|Z| \to 0$ it can be easily proved that (6.3) is bounded for $c < \sigma - m + 1$ as $t \to 0$ and $m > 0$.

This completes proof of the theorem.

Corollary — For $\sigma = m, k + m = \frac{1}{2}$ and $p = q = 1$ the transform (1.2) reduces to the transform (1.1) and our Theorem 6.1 reduces to the problem 8.5 - 1 of Zemanian (1968, p. 243).

We now define the distributional generalized Whittaker transform.

If $f(t) \in M'_{c,d}$ for $c < \sigma - m + 1$ and any value of d then the distributional generalized Whittaker transform $F(x)$ of $f(t)$ is defined by

$$F(x) = \langle f(t), \exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt) \rangle \qquad ...(6.4)$$

where the right-hand side of (6.4) has a sense as the application of $f(t) \in M'_{c,d}$ to

$$\exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt) \in M_{c,d}.$$

Theorem 6.2 (Analyticity theorem) — If

$$F(x) = \langle f(t), \exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt) \rangle$$

then $F(x)$ is a smooth function of x and

$$F^{p}(x) = \left\langle f(t), \frac{\partial^p}{\partial x^p} \exp(-\tfrac{1}{2} qxt)(xt)^{\sigma-(1/2)} W_{k,m}(pxt) \right\rangle.$$

PROOF : The proof is easy and hence omitted.

Before proving complex inversion theorem we prove the following Lemmas:

Lemma 1 — If $f \in M'_{c,d}$ and

$$\theta(x, u) = \exp(-\tfrac{1}{2} qxu)(xu)^{\sigma-(1/2)} W_{k,m}(pxu),$$

then

$$\int_0^\infty x^{-s} \langle f(u), \theta(x, u) \rangle dx = \langle f(u), \int_0^\infty \theta x^{-s} dx \rangle. \qquad ...(6.5)$$

Lemma 2 — Let $\phi \in D(I)$ and r be a fixed positive real number. If

$$P(s) = \int_0^\infty \phi(y) y^{-s} dy,$$

$s = a + iw$ and $f \in M'_{c,d}$, then

$$\int_{-r}^{r} \langle f(u), u^{s-1} \rangle P(s)\, dw = \langle f(u), \int_{-r}^{r} u^{s-1} P(s)\, dw \rangle. \qquad \ldots(6.6)$$

Lemma 3 — If

(i) $\phi \in D(I)$,

(ii) c, a and r are real numbers such that $c < \sigma - m + 1$, $a > 1$, then

$$\frac{1}{\pi} \int_0^\infty \frac{\phi(y)}{u \log\left(\frac{u}{y}\right)} \left(\frac{u}{y}\right)^a \sin\left(r \log \frac{u}{y}\right) dy$$

converges in $M_{c,d}$ to $\phi(u)$ as $r \to \infty$.

The proofs of the above Lemmas are based upon the similar Lemmas proved in Zemanian (1968, pp. 64–68) or Pathak (1979, pp. 528–29).

Theorem 6.3 (Complex inversion Theorem) — Let

(i) $f \in M'_{c,d}$,

(ii) $\phi \in D(I)$, then

$$\langle \int_0^\infty \Phi(xy)\, F(x)\, dx, \phi(y) \rangle$$

$$= \langle f, \phi \rangle \qquad \ldots(6.7)$$

where $\Phi(x)$ is given in the introduction.

PROOF : Consider

$$\langle \int_0^\infty \Phi(xy)\, F(x)\, dx, \phi(y) \rangle$$

$$= \lim_{r \to \infty} \left\langle \int_0^\infty \frac{1}{2\pi i} \int_{a-ir}^{a+ir} \frac{x^{-s} y^{-s}}{\psi(1-s)} F(x)\, dx\, ds, \phi(y) \right\rangle$$

$$= \lim_{r \to \infty} \int_0^\infty \int_0^\infty \frac{1}{2\pi i} \int_{a-ir}^{a+ir} \frac{x^{-s} y^{-s}}{\psi(1-s)} F(x)\, \phi(y)\, dx\, ds\, dy. \qquad \ldots(6.8)$$

Changing the order of integration which is permissible here and writing $s = a + iw$, (6.8)

$$= \lim_{r\to\infty} \frac{1}{2\pi} \int_{-r}^{r} \frac{1}{\psi(1-s)} \int_0^\infty x^{-s} \langle f(u), \exp(-\tfrac{1}{2} qxu)(xu)^{\sigma-(1/2)}$$

$$\times W_{k,m}(pxu) \rangle\, dx \int_0^\infty y^{-s}\phi(y)\, dy\, dw \qquad \text{...(6.9)}$$

$$= \lim_{r\to\infty} \frac{1}{2\pi} \int_{-r}^{r} \frac{1}{\psi(1-s)} \Big\langle f(u), \int_0^\infty x^{-s} \exp(-\tfrac{1}{2} qxu)(xu)^{\sigma-(1/2)}$$

$$\times W_{k,m}(pxu)\, dx \Big\rangle \int_0^\infty y^{-s}\phi(y)\, dy\, dw \qquad \text{...(6.10)}$$

(6.10) is obtained from (6.9) by using Lemma 1. By using the following result proved in (Srivastava 1968, p. 388) viz.

$$\int_0^\infty \int_0^\infty \frac{x^{-s}}{\psi(1-s)} \exp(-\tfrac{1}{2} qxu)(xu)^{\sigma-(1/2)} W_{k,m}(pxu)\, dx f(u)\, du$$

$$= \int_0^\infty u^{s-1} f(u)\, du$$

We have

(6.10)

$$= \lim_{r\to\infty} \frac{1}{2\pi} \int_{-r}^{r} \langle f(u), u^{s-1} \rangle \int_0^\infty y^{-s}\phi(y)\, dy\, dw$$

$$= \lim_{r\to\infty} \Big\langle f(u), \frac{1}{2\pi} \int_{-r}^{r} u^{s-1} \int_0^\infty y^{-s}\phi(y)\, dy\, dw \Big\rangle \qquad \text{...(6.11)}$$

(by using Lemma 2).

Changing the order of integration in (6.11) we have

$$\lim_{r\to\infty} \Big\langle f(u), \frac{1}{2\pi} \int_0^\infty \phi(y) \int_{-r}^{r} u^{s-1} y^{-s}\, dy\, dw \Big\rangle. \qquad \text{...(6.12)}$$

Now $\int_{-r}^{r} u^{s-1} y^{-s}\, dw$

$$= \int_{-r}^{r} \left(\frac{u}{y}\right)^{a} (u)^{-1} \left(\frac{u}{y}\right)^{iw} dw$$

(*equation continued on p.* 361)

$$= \int_{-r}^{r} \left(\frac{u}{y}\right)^{a} (u)^{-1} \exp\left(iw \log\left(\frac{u}{y}\right)\right)_{dw}$$

$$= 2\left(\frac{u}{y}\right)^{a} (u)^{-1} \left[\log\left(\frac{u}{y}\right)\right]^{-1} \sin\left(r \log \frac{u}{y}\right).$$

Thus, (6.12)

$$= \lim_{r\to\infty} \left\langle f(u), \frac{1}{\pi} \int_{0}^{\infty} \phi(y) \left(\frac{u}{y}\right)^{a} \sin\left(r \log \frac{u}{y}\right) \left[u \log\left(\frac{u}{y}\right)\right]^{-1} dy \right\rangle. \quad \ldots(6.13)$$

By Lemma 3, (6.13) $\to \langle f(u), \phi(u) \rangle$ as $r \to \infty$ which completes the proof of our theorem.

From the above inversion theorem the following uniqueness theorem can be deduced as a corollary.

Corollary — Let

$$F(x) = \langle f(t), \exp(-\tfrac{1}{2} qxt)\,(xt)^{\sigma-(1/2)}\, W_{k,m}(pxt) \rangle$$

for $x > \sigma_f$

(σ_f is the abscissa of convergence).

and

$$G(x) = \langle g(t), \exp(-\tfrac{1}{2} qxt)\,(xt)^{\sigma-(1/2)}\, W_{k,m}(pxt) \rangle$$

for $x > \sigma_g$.

Then in the sense of equality in $D'(I)$, $f = g$.

ACKNOWLEDGEMENT

The author expresses his gratefulness to the referee for his constructive criticisms.

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