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ON CONVOLUTION INTEGRAL EQUATIONS

R. K. RAINA

Department of Mathematics, S. K. N. Agriculture College, (University of Udaipur) Jobner Campus, Rajasthan 303329

AND

C. L. KOUL

Department of Mathematics, M. R. Engineering College, Jaipur, Rajasthan 302004

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A double convolution integral equation having an H-function of two variables as its kernel is solved. The solutions of four other double convolution equations are derived as special cases of the main result.

1. INTRODUCTION

Srivastava and Buschman (1974) solved a convolution integral equation with Fox's H-function as its kernel. This work was later extended by Buschman *et al.* (1976), who obtained the solution of a convolution integral equation with the H-function of two variables as its kernel. Here we consider the double convolution integral equation

$$\int_0^x \int_0^y (x-\xi)^{\sigma-1} (y-\eta)^{\rho-1} H_1 [x-\xi, y-\eta]\, f(\xi, \eta)\, d\xi\, d\eta = g(x, y) \qquad \ldots(1)$$

where $\quad \text{Re}(\sigma) > 0, \quad \text{Re}(\rho) > 0.$

The H_1-function occurring in (1) is a particular case of the general H-function of two variables, defined by Mittal and Gupta (1972) which possesses the following integral representation:

$$H_1[x, y] = -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi(s, t)\, \Gamma(-s)\, \Gamma(-t)\, \theta_1(s)\, \theta_2(t)\, x^s y^t\, ds\, dt \qquad \ldots(2)$$

where

$$\phi(s, t) = [\prod_{j=1}^{P} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{Q} \Gamma(1 - b_j + \beta_j s + B_j t)]^{-1} \qquad \ldots(3)$$

and

$$\theta_1(s) = \prod_{j=1}^{m} \Gamma(1 - c_j + \gamma_j s) \, . \, [\prod_{j=1}^{T} \Gamma(1 - d_j + \delta_j s)$$

$$\times \prod_{j=m+1}^{S} \Gamma(c_j - \gamma_j s)]^{-1} \qquad \ldots(4)$$

$\theta_2(t)$ stands for the Gamma quotients defined analogously to (4), in terms of the parameter sets $(e_j, E_j)_{1,U}$, $(f_j, F_j)_{1,V}$. An empty product is interpreted as unity and all of Greek and the Capital letters on the right side of (3) and (4) are assumed to be positive. We adopt the following notation for the function H_1, in which the parameters are analogously displayed to the notation introduced by Srivastava and Panda (1976) who were influenced by the notations used earlier by Srivastava and Joshi (1969):

$$H_1[x, y] = H^{0,0:1,m;1,n}_{P,Q:S,T+1;U,V+1} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} ((a_P; \alpha_P, A_P)) : ((c_S, \gamma_S)); ((e_U, E_U)) \\ ((b_Q; \beta_Q, B_Q)) : (0, 1), ((d_T, \delta_T)); (0, 1), ((f_V, F_V)) \end{matrix} \right] \qquad \ldots(5)$$

where $((a_P; \alpha_P, A_P))$ denotes the sequence of P-parameters $(a_1; \alpha_1, A_1), \ldots, (a_P; \alpha_P, A_P)$, etc.

2. Solution of the Integral Equation

To solve the double convolution integral eqn. (1), we take the double Laplace transform, given by

$$\bar{f}(p, q) = \int_0^\infty \int_0^\infty e^{-px-qy} f(x, y) \, dx \, dy, \ \text{Re}(p) > 0, \ \text{Re}(q) > 0, \qquad \ldots(6)$$

of both sides of eqn. (1), use the two-dimensional generalization of the well-known theorem on convolution property in one dimension and the result of Raina (1975, p. 140) and then rewrite the equation in the following form:

$$\bar{f}(p, q) = p^\sigma q^\rho \bar{g}(p, q) \, (H_2 [p^{-1}, q^{-1}])^{-1} \qquad \ldots(7)$$

where $\bar{f}(p, q)$ denotes the Laplace transform of $f(x, y)$ and

$$H_2 [p^{-1}, q^{-1}] = H^{0,0:1,m+1:1,n+1}_{P,Q:S+1,T+1;U+1,V+1} \left[\begin{matrix} p^{-1} \\ q^{-1} \end{matrix} \middle| \begin{matrix} ((a_P; \alpha_P, A_P)) : (1 - \sigma, 1), ((c_S, \gamma_S)); (1 - \rho, 1), ((e_U, E_U)) \\ ((b_Q; \beta_Q, B_Q)) : (0, 1), ((d_T, \delta_T)); (0, 1), ((f_V, F_V)) \end{matrix} \right]. \qquad \ldots(8)$$

Using Goyal (1975, p. 119), we obtain

$$H_2\,[p^{-1}, q^{-1}] = \sum_{M,N=0}^{\infty} \frac{\Gamma(\sigma + M)\,\Gamma(\rho + N)\,(-1)^{M+N}}{M!N!} \cdot A_{M,N} p^{-M} q^{-N} \qquad \ldots(9)$$

where

$$A_{M,N} = \phi(M, N)\,\theta_1(M)\,\theta_2(N) \qquad \ldots(10)$$

ϕ, θ being defined in (3) and (4).

Equation (9) is further written as

$$H_2\,[p^{-1}, q^{-1}] = \sum_{M,N=0}^{\infty} A^*_{M,N}\, p^{-M} q^{-N} \qquad \ldots(11)$$

where

$$A^*_{M,N} = \frac{\Gamma(\sigma + M)\,\Gamma(\rho + N)\,(-1)^{M+N}}{M!N!} \cdot A_{M,N}. \qquad \ldots(12)$$

To obtain the reciprocal of $H_2\,[p^{-1}, q^{-1}]$, occurring in (7), we assume that

$$(H_2\,[p^{-1}, q^{-1}])^{-1} = \sum_{\lambda,\mu=0}^{\infty} H_{\lambda,\mu}\, p^{-\lambda} q^{-\mu}. \qquad \ldots(13)$$

(11) in conjunction with (13) then yields

$$\left(\sum_{\lambda,\mu=0}^{\infty} H_{\lambda,\mu}\, p^{-\lambda} q^{-\mu}\right)\left(\sum_{M,N=0}^{\infty} A^*_{M,N}\, p^{-M} q^{-N}\right) = 1. \qquad \ldots(14)$$

On comparing the coefficients of like powers on the two sides of (14), we find that the coefficients $H_{\lambda,\mu}$ can be uniquely determined from the relations

$$\begin{cases} H_{0,0} \cdot A^*_{0,0} = 1 \\ \displaystyle\sum_{r=0}^{u} \sum_{s=0}^{v} H_{u-r,s} \cdot A^*_{r,v-s} = 0 \\ (u, v = 0, 1, 2, \ldots;\ u \geqslant r,\ v \geqslant s). \end{cases} \qquad \ldots(15)$$

Now (7) with the help of (13) becomes

$$\bar{f}(p, q) = p^{\sigma} q^{\rho} \left(\sum_{\lambda,\mu=0}^{\infty} H_{\lambda,\mu}\, p^{-\lambda} q^{-\mu}\right) \bar{g}(p, q)$$

$$= p^{-(a-\sigma)}\, q^{-(b-\rho)} \left(\sum_{\lambda,\mu=0}^{\infty} H_{\lambda,\mu}\, p^{-\lambda} q^{-\mu}\right) (p^{a} q^{b} \bar{g}(p, q)). \qquad \ldots(16)$$

Applying inversion and convolution [see Ditkin and Prudnikov (1962, pp. 93, 137), we arrive at the following:

Theorem — Let

(i) $a > \text{Re}(\sigma) > 0, b > \text{Re}(\rho) > 0,$

(ii) $\bar{g}_{1,y^l}(p, 0) = 0 (0 \leqslant l \leqslant b - 1)$, $\bar{g}_{2,x^k}(0, q) = 0 (0 \leqslant k \leqslant a - 1)$

$$g^{(k+l)}_{x^k, y^l}(0, 0) = 0 (0 \leqslant l \leqslant b, 0 \leqslant k \leqslant a),$$

(iii) m, n, P, Q, S, T, U, V be non-negative integers and the Greek and the capitalized parameters be positive,

(iv)
$$\sum_{j=1}^{P} \alpha_j + \sum_{j=1}^{S} \gamma_j < \sum_{j=1}^{Q} \beta_j + \sum_{j=1}^{T} \delta_j + 1,$$

$$\sum_{j=1}^{P} A_j + \sum_{j=1}^{U} E_j < \sum_{j=1}^{Q} B_j + \sum_{j=1}^{V} F_j + 1,$$

$$\sum_{j=1}^{m} \gamma_j - \sum_{j=m+1}^{S} \gamma_j - \sum_{j=1}^{P} \alpha_j - \sum_{j=1}^{Q} \beta_j - \sum_{j=1}^{T} \delta_j + 1 > 0,$$

$$\sum_{j=1}^{n} E_j - \sum_{j=n+1}^{U} E_j - \sum_{j=1}^{P} A_j - \sum_{j=1}^{Q} B_j - \sum_{j=1}^{V} F_j + 1 > 0.$$

Then the solution of (1) is given by

$$f(\xi, \eta) = \int_0^{\xi} \int_0^{\eta} (\xi - x)^{a-\sigma-1} (\eta - y)^{b-\rho-1} W[\xi - x, \eta - y]$$

$$\times g^{(a+b)}_{x^a, y^b}(x, y)\, dx\, dy, \qquad \ldots(17)$$

where

$$W[x, y] = \sum_{\lambda,\mu=0}^{\infty} \frac{H_{\lambda,\mu}}{\Gamma(a - \sigma + \lambda)\, \Gamma(b - \rho + \mu)} x^{\lambda} y^{\mu}, \qquad \ldots(18)$$

$H_{\lambda,\mu}$ being given by (15),

$$\bar{g}_{1,y^l}(p, 0) = \int_0^{\infty} e^{-p\xi} \frac{\partial^l}{\partial y^l} g(\xi, y) \Big|_{y=0} d\xi \qquad \ldots(19)$$

$$\bar{g}_{2,x^k}(0, q) = \int_0^{\infty} e^{-q\eta} \frac{\partial^k}{\partial x^k} g(x, \eta) \Big|_{x=0} d\eta \qquad \ldots(20)$$

and $\qquad g^{(k+l)}_{x^k, y^l(x,y)}$ stands for $\dfrac{\partial^{k+l}}{\partial x^k \partial y^l} g(x, y)$.

3. Special Cases

The H-function of two variables occurring in (1) has, as its particular cases, a number of special functions (see, Mittal and Gupta (1972)) and so a number of special cases of the aforementioned theorem can be given. For example, if we put $P = Q = 0$ in (1), the above theorem reduces to the following form involving the Fox's H-functions:

Corollary 1 — The double convolution equation

$$\int_0^x \int_0^y (x-\xi)^{\sigma-1} (y-\eta)^{\rho-1} H^{1,m}_{S,T+1}\left[x-\xi \,\middle|\, \begin{matrix} ((c_S, \gamma_S)) \\ (0, 1), ((d_T, \delta_T)) \end{matrix}\right]$$

$$\times H^{1,n}_{U,V+1}\left[y-\eta \,\middle|\, \begin{matrix} ((e_U, E_U)) \\ (0, 1), ((f_V, F_V)) \end{matrix}\right] f(\xi,\eta)\, d\xi\, d\eta = g(x, y) \qquad \ldots(21)$$

possesses the solution

$$f(\xi, \eta) = \int_0^\xi \int_0^\eta (\xi - x)^{a-\sigma-1} (\eta - y)^{b-\rho-1} \nu[\xi - x, \eta - y]\, g^{(a+b)}_{x^a, y^b}(x, y)\, dx\, dy, \qquad \ldots(22)$$

under suitable restrictions on the parameters, provided that conditions (i) and (ii) of the main theorem are satisfied, where

$$\nu[x, y] = \sum_{\lambda,\mu=0}^{\infty} G_{\lambda,\mu} \frac{x^\lambda y^\mu}{\Gamma(a - \sigma + \lambda)\, \Gamma(b - \rho + \mu)} \qquad \ldots(23)$$

the coefficients $G_{\lambda,\mu}$ are suitably derived from $H_{\lambda,\mu}$ (given by (15)) by putting $P = Q = 0$, therein, with

$$A_{M,N} = \theta_1(M)\, \theta_2(N) \qquad \ldots(24)$$

and $A^*_{M,N}$ given by (12).

If we take all γ's, δ's, E's F's equal to unity and set $m = S = T = n = U = V = 1$, and make other slight changes in the parameters in Corollary 1, it yields:

Corollary 2 — The integral equation

$$\int_0^x \int_0^y (x - \xi)^{\sigma-1} (y - \eta)^{\rho-1} {}_1F_1(c; d; -(x - \xi))\, {}_1F_1(e; f; -(y - \eta))$$

$$\times f(\xi, \eta)\, d\xi\, d\eta = g(x, y) \cdot \Gamma(d)\, \Gamma(f)\, [\Gamma(c)\, \Gamma(e)]^{-1} \qquad \ldots(25)$$

has the solution

$$f(\xi, \eta) = \int_0^\xi \int_0^\eta (\xi - x)^{a-\sigma-1} (\eta - y)^{b-\rho-1} A\,[\xi - x, \eta - y]\, g_{x^a, y^b}^{(a+b)}(x, y)\, dx\, dy, \quad \ldots(26)$$

where

$$A\,[x, y] = \sum_{\lambda,\mu=0}^{\infty} K_{\lambda,\mu} \frac{x^\lambda y^\mu}{\Gamma(a - \sigma + \lambda)\,\Gamma(b - \rho + \mu)}, \quad \ldots(27)$$

provided that the conditions (i) and (ii) of the theorem are satisfied.

The coefficients $K_{\lambda,\mu}$ in (27) are given by

$$\begin{cases} K_{0,0} \cdot A^*_{0,0} = 1, \quad \sum_{r=0}^{u} \sum_{s=0}^{v} K_{u-r,s} \cdot A^*_{r,v-s} = 0, \\ (u, v = 0, 1, 2, \ldots;\ u \geqslant r,\ v \geqslant s), \end{cases} \quad \ldots(28)$$

where

$$A^*_{M,N} = \frac{(-1)^{M+N}\Gamma(\sigma + M)\,\Gamma(\rho + N)\,\Gamma(c + M)\,\Gamma(e + N)}{M!N!\Gamma(d + M)\,\Gamma(f + N)}. \quad \ldots(29)$$

If, in corollary 2, we put $\sigma = d$, $\rho = f$, we find that $A\,[x, y]$ occurring in (27) turns out to be a product of two ${}_1F_1$' s.

The result obtained is:

Corollary 3 — The double convolution equation

$$\int_0^x \int_0^y (x - \xi)^{d-1} (y - \eta)^{f-1}\, {}_1F_1(c; d; -(x - \xi))\, {}_1F_1(e; f; -(y - \eta))\, f(\xi, \eta)\, d\xi\, d\eta$$

$$= g(x, y) \cdot \Gamma(d)\,\Gamma(f)\,[\Gamma(c)\,\Gamma(e)]^{-1}, \quad \ldots(30)$$

has the solution given by

$$f(\xi, \eta) = [\Gamma(a - d)\,\Gamma(b - f)\,\Gamma(c)\,\Gamma(e)]^{-1} \cdot \int_0^\xi \int_0^\eta (\xi - x)^{a-d-1} (\eta - y)^{b-f-1}$$

$$\times\, {}_1F_1(-c; a - d; -(\xi - x))\, {}_1F_1(-e; b - f; -(\eta - y))\, g_{x^a, y^b}^{(a+b)}(x, y)\, dx\, dy, \quad \ldots(31)$$

provided that

$a > \text{Re}\,(d) > 0$, $b > \text{Re}\,(f) > 0$, and the conditions (ii) given in the main theorem are satisfied.

In case we reduce the H-function of two variables occurring in (1) to the Appell's function F_3 by setting $P = T = V = 0$, $m = n = S = U = 2$ and taking $b_1 = 1-\gamma$,

$c_1 = 1 - \alpha$, $c_2 = 1 - \beta$, $e_1 = 1 - \alpha'$, $e_2 = 1 - \beta'$, $\beta_1 = \gamma_1 = \gamma_2 = B_1 = E_1 = E_2 = 1$ therein, then Theorem 1 yields the following result [see also Srivastava (1976) who solved a convolution integral equation involving the confluent hypergeometric function Φ_2^r of r complex variables] :

Corollary 4 — The convolution equation

$$A \int_0^x \int_0^y (x - \xi)^{\sigma-1} (y - \eta)^{\rho-1} F_3(\alpha, \alpha', \beta, \beta'; \gamma; -(x - \xi), -(y - \eta))$$

$$\times f(\xi, \eta)\, d\xi\, d\eta = g(x, y) \qquad ...(32)$$

has for solution

$$f(\xi, \eta) = \int_0^\xi \int_0^\eta (\xi - x)^{a-\sigma-1} (\eta - y)^{b-\rho-1} W_1 [\xi - x, \eta - y]$$

$$\times g^{a+b}_{x^a, y^b} (x, y)\, dx\, dy \qquad ...(33)$$

where

$$W_1 [x, y] = \sum_{\lambda,\mu=0}^{\infty} \frac{L_{\lambda,\mu}}{\Gamma(a - \sigma + \lambda)\, \Gamma(b - \rho + \mu)} x^\lambda y^\mu \qquad ...(34)$$

and

$$\begin{cases} L_{0,0} \cdot B^*_{0,0} = 1 \\ \sum_{r=0}^{u} \sum_{s=0}^{v} L_{u-r,s}\, B^*_{r,v-s} = 0, \ (u, v = 0, 1, 2, \ldots;\ u \geqslant r,\ v \geqslant s) \end{cases} \qquad ...(35)$$

with $$B^*_{M,N} = \frac{(-1)^{M+N}}{M!N!} \frac{(\sigma)_M (\rho)_N (\alpha)_M (\beta)_M (\alpha')_N (\beta')_N}{(\gamma)_{M+N}} \cdot B,$$

where $B = \Gamma(\sigma)\, \Gamma(\rho)\, A$,

$A = \Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\alpha')\, \Gamma(\beta')/\Gamma(\gamma)$

provided that conditions (i) and (ii) of Theorem 1 are satisfied.

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