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SPIRAL-LIKE FUNCTIONS WITH FIXED SECOND COEFFICIENT

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Let $F_p(\alpha, \beta)$ denote the class of functions $f(z)$ which are regular in the unit disk E and of the form $f(z) = z + |a_2| e^{-i\alpha} z^2 + \dots$, where $|a_2| = 2p(1 - \beta) \cos \alpha$ and satisfy the condition

$$\text{Re } e^{i\alpha} z \frac{f'(z)}{f(z)} > \beta \cos \alpha \text{ for all } z \text{ in } E.$$

In this paper we have found the sharp radius of γ-spiralness of the functions belonging to the class $F_p(\alpha, \beta)$.

§1. Let $\mathcal{A}$ denote the class functions $f(z)$ which are regular and univalent in the unit disk $E = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0 = f'(0) - 1$.

Let $F(\alpha, \beta)$ denote the class of functions $f(z) = z + a_2 z^2 + \dots$ which satisfy the condition

$$\text{Re } e^{i\alpha} \frac{zf'(z)}{f(z)} > \beta \cos \alpha \quad \text{for all } z \text{ in } E. \qquad \dots(1.1)$$

The class of functions $F(\alpha, \beta)$ called the class of α-spiral functions of order β was introduced and studied by Libera (1967).

Libera (1967) proved that if $f(z) = z + a_2 z^2 + \dots, \epsilon F(\alpha, \beta)$ then,

$$|a_2| \leqslant 2(1 - \beta) \cos \alpha. \qquad \dots(1.2)$$

If $\epsilon = \exp(-i \arg a_2 - i\alpha)$, then $\frac{f(\epsilon z)}{\epsilon} = z + |a_2| e^{-i\alpha} z^2 + \dots, \epsilon F(\alpha, \beta)$ whenever $f(z) \in F(\alpha, \beta)$. Thus without loss of generality we can replace the second coefficient a_2 of $f(z) \in F(\alpha, \beta)$ by $|a_2| \cdot e^{-i\alpha}$.

Let $F_p(\alpha, \beta)$ denote the class of functions $f(z) = z + |a_2| e^{-i\alpha} z^2 + \dots$, which satisfy (1.1), where $|a_2| = 2p(1 - \beta) \cos \alpha$. In view of (1.2) it follows that $0 \leqslant p \leqslant 1$.

Let $G_p(\alpha, \beta)$ denote the class of functions $g(z) = z + |b_2| e^{-i\alpha} z^2 + \dots$, regular in E and satisfy the condition

$$\text{Re } e^{i\alpha}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \beta \cos \alpha, \quad z \in E, \qquad \dots(1.3)$$

where $\quad | b_2 | = p(1 - \beta) \cos \alpha.$

Clearly $\quad g(z) \in G_p(\alpha, \beta)$ iff $zg'(z) \in F_p(\alpha, \beta).$

In this paper we determine the sharp radius of γ-spiralness of the functions belonging to the class $F_p(\alpha, \beta)$, generalizing an earlier result due to Libera (1967) and Umarani (1976).

The technique employed to obtain this result is similar to that used by McCarty (1972).

§2. *Lemma* 1 — If $f(z) \in F_p(\alpha, \beta)$, then

$$\left| \frac{zf'(z)}{f(z)} - w_0 \right| \leqslant \rho_0, \qquad \ldots(2.1)$$

where
$$w_0 = \frac{(1 + pr)^2 + [(1 - 2\beta) \cos \alpha - i \sin \alpha]\, e^{-i\alpha} \;\; r^2(r + p)^2}{(1 - r^2)(1 + 2pr + r^2)} \qquad \ldots(2.2)$$

and

$$\rho_0 = \frac{2(1 - \beta) \cos \;\alpha r(1 + pr)(r + p)}{(1 - r^2)(1 + 2pr + r^2)} \qquad \ldots(2.3)$$

The result is sharp.

PROOF : Let $f(z) \in F_p(\alpha, \beta)$. Then there exists a function $w(z)$ analytic in E and $| w(z) | < 1$ in E such that

$$e^{i\alpha} \frac{zf'(z)}{f(z)} = \cos \alpha \frac{1 + (1 - 2\beta) w(z)}{1 - w(z)} + i \sin \alpha$$

or

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(1 - 2\beta) \cos \alpha - i \sin \alpha]\, e^{-i\alpha} w(z)}{1 - w(z)}.$$

Solving for $w(z)$,

$$w(z) = \frac{\{zf'(z)/f(z)\} - 1}{\{zf'(z)/f(z)\} + [(1 - 2\beta) \cos \alpha - i \sin \alpha]\, e^{-i\alpha}}$$

Since $f(z) = z + | a_2 | \;\; e^{-i\alpha} z^2 + \ldots$, we obtain $w(z) = zp + \ldots = z\phi(z)$, where $\phi(z)$ is analytic in E, $\phi(0) = p$ and $| \phi(z) | \leqslant 1$ in E.

Now $\quad \dfrac{\phi(z) - p}{1 - p\phi(z)} \prec z.$ Therefore $\phi(z) \prec \dfrac{z + p}{1 + p(z)}$

Also, $\quad | w(z) | = | z \,\phi(z) | \leqslant \dfrac{| z | + p}{| + p.|z|} \cdot | z |$

Let $\quad g(z) = \dfrac{| z | + p}{1 + p \;| z |} \; z$

and

$$h(z) = \frac{1 + [(1 - 2\beta) \cos \alpha - i \sin \alpha]\, e^{-i\alpha} z}{1 - z}.$$

Since the image of $|z| \leqslant r$ under $g(z)$ is a disk and $h(z)$ is a bilinear transformation. Hence $zf'(z)/f(z)$ is subordinate to (hog) (z). That is, the image of $|z| \leqslant r$ under $zf'(z)/f(z)$ is contained in the image of $|z| \leqslant r$ under (hog) (z).

Equality in (2.1) can be attained by a function

$$f(z) = \frac{z}{(1 - 2pz + z^2)}{}^{(1 - \beta) \cos \alpha e^{-i\alpha}} \qquad \ldots(2.4)$$

$$= z + 2p(1 - \beta) \cos \alpha e^{-i\alpha} z^2 + \ldots .$$

Hence

$$\frac{zf'(z)}{f(z)} = \frac{1 - 2pz + z^2 - 2(1-\beta) \cos \alpha e^{-i\alpha}\, z(z - p)(1 - pz)}{1 - 2pz + z^2}$$

$$= \frac{1 + \psi - 2(1 - \beta) \cos\alpha e^{-i\alpha}\, \psi}{1 + \psi} \qquad \ldots(2.5)$$

where $\psi = \dfrac{z(z - p)}{1 - pz}$.

Since $p \leqslant 1$, $|\psi| < 1$ for $z \in E$.

This shows that

$$\text{Re}\, e^{i\alpha} \frac{zf'(z)}{f(z)} = \text{Re} \left\{ \cos \alpha \frac{1 - (1 - 2\beta)\psi}{1 + \psi} + i \sin \alpha \right\} > \beta \cos \alpha.$$

Hence $f(z) \in F_p(\alpha, \beta)$.

Substituting $\psi = -\dfrac{\delta(\delta - e^{i\alpha})}{1 - e^{i\alpha}\delta}$, where

$\delta = \dfrac{r(r + p)}{1 + rp}$ in (2.5), we find that $\left| \dfrac{zf'(z)}{f(z)} - w_0 \right| = \rho_0$, where w_0 and ρ_0 are given by (2.2) and (2.3).

This completes the proof of the Lemma.

If $\alpha = 0$ in Lemma 1, we obtain a result of McCarty (1972).

Theorem 1 — If $f(z) \in F_p(\alpha, \beta)$, then $f(z)$ is γ-spiral for $|z| < r_\gamma$, where r_γ is the smallest $+ve$ root of the equation.

$$\cos \gamma + 2p\, [\cos \gamma - (1 - \beta) \cos \alpha]\, r + [p^2 \cos \gamma + c.p^2$$
$$- 2(1 - \beta) \cos \alpha (1 + p^2)]\, r^2$$
$$+ 2p[c - (1 - \beta) \cos \alpha]\, r^3 + c.r^4 = 0, \qquad \ldots(2.6)$$

where $c = \cos(\gamma - 2\alpha) - 2\beta \cos(\gamma - \alpha)$.
The result is sharp.

PROOF: Let $f(z) \in F_p(\alpha, \beta)$. Then by the above Lemma, we have

$$\left| \frac{zf'(z)}{f(z)} - w_0 \right| \leqslant \rho_0.$$

Hence
$$\text{Re } e^{i\gamma} \frac{zf'(z)}{f(z)} \geqslant \text{Re } e^{i\gamma} w_0 - \rho_0$$

$$= \frac{\cos\gamma(1 + pr)^2 + [\cos(\gamma - 2\alpha) - 2\beta \cos\alpha \cos(\gamma - \alpha)] r^2 (r + p)^2 - 2(1 - \beta) \cos\alpha r(1 + pr)(r + p)}{(1 - r^2)(1 + 2pr + r^2)}. \quad \ldots(2.7)$$

$f(z)$ is γ-spiral if the R.H.S. of (2.7) is positive. Hence $f(z)$ is γ-spiral for $|z| < r\gamma$ where $r\gamma$ is the smallest positive root of the equation

$$\cos\gamma(1 + pr)^2 + [\cos(\gamma - 2\alpha) - 2\beta \cos\alpha \cos(\gamma - \alpha)] r^2 (r + p)^2$$
$$- 2(1 - \beta) \cos\alpha r(1 + pr)(r + p) = 0.$$

Simplifying the above equation, we obtain (2.6).

Sharpness follows from the fact that there is a function for which equality can be attained in (2.1).

If $\gamma = 0$ in the above theorem we obtain the radius of starlikeness of the class $F_p(\alpha, \beta)$.

Corollary 1 — $f(z) \in F_p(\alpha, \beta)$ is starlike for $|z| < r_0$, where r_0 is the least positive root of the equation

$$1 + 2p(1 - (1 - \beta)\cos\alpha) r + 2(1 - \beta)\cos\alpha [\cos\alpha p^2 - 1 - p^2] r^2$$
$$+ 2p(c - (1 - \beta)\cos\alpha) r^3 + cr^4 = 0,$$

where $c = 2(1 - \beta)\cos^2\alpha - 1$. ...(2.8)

If $p = 1$ in Theorem 1, we obtain a result of Libera (1967) which is also obtained by the author (Umarani 1976) using a different method.

Corollary 2 — $f(z) \in F(\alpha, \beta)$ is γ-spiral for $|z| < r\gamma$, where $r\gamma$ is the least $+ve$ root of the equation

$$\cos\gamma - 2(1 - \beta)\cos\alpha r + [-2\beta\cos(\gamma - \alpha)\cos\alpha + \cos(\gamma - 2\alpha)] r^2 = 0.$$

If $p = 1$, $\beta = 0$ and $\gamma = 0$ in Theorem 1, we obtain the radius of starlikeness of spiral-like functions which is a result due to Robertson (1965).

Since $g(z) \in G_p(\alpha, \beta)$ iff $zg'(z) \in F_p(\alpha, \beta)$, we obtain from Theorem 1.

Theorem 2 — If $g(z) \in G_p(\alpha, \beta)$ then

$$\text{Re } e^{i\gamma}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0 \text{ for } |z| < r\gamma,$$ where $r\gamma$ is the least $+ve$ root of eqn. (2.6).

The result is sharp.

If $\gamma = 0$ in Theorem 2, we obtain the radius of convexity of the class $G_p(\alpha, \beta)$.

Corollary 3 — If $g(z) \in G_p(\alpha, \beta)$, then the radius of convexity of $g(z)$ is the least $+ve$ root of eqn. (2.8).

For $p = 1$, Theorem 2 generalizes the result of the author (Umarani 1976).

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