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TORSIONAL VIBRATIONS OF FINITE COMPOSITE POROELASTIC CYLINDERS

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Employing Biot's theory for wave propagation in porous solids, the frequency equation for torsional vibrations of composite poroelastic cylinders, either concentric or bonded end to end is obtained. The variation offrequency is displayed graphically. The results of classical theory are obtained as a particular case.

1. INTRODUCTION

The basic formulation for propagation of stress waves in liquid-filled porous media is due to Biot (1956), whose model consists of an elastic matrix permeated by a network of interconnected spaces saturated with liquid. An account of further researches based on Biot's work is given by Paria (1963). Using this theory, we the author studied some problems of torsional vibrations. The purpose of the present paper is to study the torsional vibrations of finite composite poroelastic cylinders. (See Tajuddin 1978, Tajuddin and Sarma 1978a,b, 1980).

A composite poroelastic cylinder can be formed either by joining two cylinders of different materials at plane ends or by joining concentric cylinders to have a common curved surface. In this paper, torsional vibrations of two different types of finite composite poroelastic circular cylinders is studied namely (i) a hollow composite poroelastic cylinder with two concentric cylindrical layers having a common curved surface and (ii) a solid composite poroelastic cylinder bonded end to end. The frequency equation in each case is obtained. In the former case, the variation of frequency with inner cylinder thickness is studied for the lowest symmetric and antisymmetric modes. In the latter case, the variation of frequency with change in ratio of lengths of cylinders for the least mode of vibration is presented. By neglecting liquid effects, the results classical theory are recovered.

2. BASIC EQUATIONS

In the case of torsional vibrations, the non-zero stress components in terms of displacement are

$$\sigma_{r\theta} = Nr\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \qquad \ldots(2.1)$$

$$\sigma_{\theta z} = N\frac{\partial u_\theta}{\partial z} \cdot$$

The non-trivial stress equations of motion without body forces are

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = (\rho_{11}\ddot{u}_\theta + \rho_{12}\ddot{U}_\theta)$$

$$\text{grad } \sigma = (\rho_{12}\ddot{u}_\theta + \rho_{22}\ddot{U}_\theta) \qquad \text{...(2.2)}$$

where a 'dot' over a quantity indicates differential coefficient with respect to time, t. The remaining symbols are the same as used by Biot (1956). The dilatations of solid and liquid are zero, so the excess pore-pressure (σ) developed in solid-liquid aggregate is zerro.

3. Torsional Vibrations of a Finite Hollow Composite Poroelastic Cylinder

Torsional vibrations in a finite hollow concentric composite poroelastic cylinder of different materials are considered. Let r, θ, z be cylindrical polar coordinates with z-axis along the axis of the cylinder. Let the materials be homogeneous, isotropic and the length of cylinder be $2c$. Assume the radii of outer and inner core be r_1 and r_2 and interface is $r = a$. In this case, the non-zero displacement is

$$u_\theta = v(r) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (kz) \exp(ipt), \quad U_\theta = V(r) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (kz) \exp(ipt). \qquad \text{...(3.1)}$$

cos (kz) or sin (kz) is taken according as the motion is symmetric or antisymmetric about the central plane. Substitution of eqn. (3.1) into equations of motion (2.2) gives

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \left(q^2 - \frac{1}{r^2}\right) v = 0 \qquad \text{...(3.2)}$$

where

$$q^2 = s^2 - k^2, \; s^2 = \frac{p^2(\rho_{11}\rho_{22} - \rho_{12}^2)}{N\rho_{22}}. \qquad \text{...(3.3)}$$

From eqns. (3.1) and (3.2), one obtains

$$u_{\theta j} = [c_{1j} J_1(q_j r) + c_{2j} Y_1(q_j r)] \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (k_j z) \exp(ipt), \; (j = 1, 2). \qquad \text{...(3.4)}$$

$j = 1, 2$ refer to the quantities of outer and inner cylinders.

The pore-pressure vanishes identically. The boundary conditions are:

On the curved surface

$$\begin{aligned} \text{At} \quad & r = r_1 \quad (\sigma_{r\theta})_1 = 0, \quad k_1 = k_2 \\ & r = r_2 \quad (\sigma_{r\theta})_2 = 0, \\ & r = a \quad (\sigma_{r\theta})_1 = (\sigma_{r\theta})_2, \; (u_\theta)_1 = (u_\theta)_2, \end{aligned} \qquad \text{...(3.5)}$$

On the plane ends

$$\sigma_{\theta z} = 0 \text{ for } z = \pm c. \qquad \text{...(3.6)}$$

The boundary conditions on the curved surface gives four homogeneous equations in four unknowns c_{11}, c_{21}, c_{12}, c_{22}.

The requirement for a non-trivial solution yields the frequency equation to be

$$\left.\begin{aligned} & - N_2 q_2 \{-J_2(q_1 r_1) Y_1(q_1 a) + Y_2(q_1 r_1) J_1(q_1 a)\} \\ & \times \{J_2(q_2 r_2) Y_2(q_2 a) - Y_2(q_2 r_2) J_2(q_2 a)\} \\ & + N_1 q_1 \{J_2(q_1 r_1) Y_2(q_1 a) - Y_2(q_1 r_1) J_2(q_1 a)\} \\ & \times \{Y_2(q_2 r_2) J_1(q_2 a) - J_2(q_2 r_2) Y_1(q_2 a)\} = 0. \end{aligned}\right\} \qquad \text{...(3.7)}$$

Let the outer casing of the cylinder be very thin and its stiffness large compared to that of core material, that is $r_1/a \to 1$ and $N_1/N_2 \to \infty$, respectively. With these assumptions, the frequency eqn. (3.7) becomes indeterminate at $r_1 = a$. Hence, by taking Taylor's series expansion in the neighbourhood of $r_1/a = 1$ and letting $N_1 = N_2\, ((r_1/a) - 1)^{-1}$, the frequency eqn. (3.7) will be modified to

$$\begin{aligned} & [Y_2(q_2 r_2) J_1(q_2 a) - Y_1(q_2 a) J_2(q_2 r_2)] + q_2 k^{-2} a^{-1} [J_2(q_2 r_2) Y_2(q_2 a) \\ & \quad - Y_2(q_2 r_2) J_1(q_2 a)] = 0. \end{aligned} \qquad \text{...(3.8)}$$

The boundary conditions on the flat ends give sin $(kc) = 0$ or cos $(kc) = 0$ according as the motion is symmetric or antisymmetric. This is satisfied only when

$$kc = n_1 \pi,\ kc = (2n_2 + 1)\frac{\pi}{2}, \qquad \text{...(3.9)}$$

where n_1 and n_2 are positive integers.

The frequency eqn. (3.8) can be rewritten in non-dimensional form as

$$\begin{aligned} & [Y_2(xy) J_1(x) - Y_1(x) J_2(xy)] + k^{-2} c^{-2} (a^2/c^2) [J_2(xy) Y_2(x) \\ & \qquad - Y_2(xy) J_1(x)] = 0 \end{aligned} \qquad \text{...(3.10)}$$

where

$$m_1 = \frac{\rho_{11}}{\rho},\ m_2 = \frac{\rho_{12}}{\rho},\ m_3 = \frac{\rho_{22}}{\rho},\ f = \frac{ap}{c_0}$$

$$m_4 = \frac{m_1 m_3 - m_2^2}{m_3},\ x = \left(f^2 m_4^{(2)} - k^2 c^2 \left(\frac{a^2}{c^2}\right)\right)^{1/2} y = \frac{r_2}{a}$$

and

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22},\ c_0^2 = N/\rho. \qquad \text{...(3.11)}$$

By neglecting liquid effects in eqns. (3.10), the frequency equation of classical theory is recovered.

The non-dimensional frequency eqn. (3.10) can be solved, f as a function of y, for a particular cylinder and for a particular mode. Let the dimensions of the cylinder (diameter-thickness ratio a/c) be 0.1, 0.2, 0.5 and 1. The value of kc required in eqn. (3.11) is determined from eqn. (3.9), according as the motion is symmetric and antisymmetric. Let these values be $(n_1 =)1$ and $(n_2 =)0$, respectively and material constants required in the problem are considered to be

$$m_1 = 0.8, m_2 = 0, m_3 = 0.2.$$

The variation of frequency (f) with inner radius ratio (y) is presented graphically in Fig. 1 for symmetric and antisymmetric motion.

From Fig. 1, it is observed that for symmetric motion, the value of frequency upto inner radius ratio is equal to 0.6 remains almost constant except for diameter thickness ratio is equal to 1. For diameter thickness ratio is equal to 1, frequency

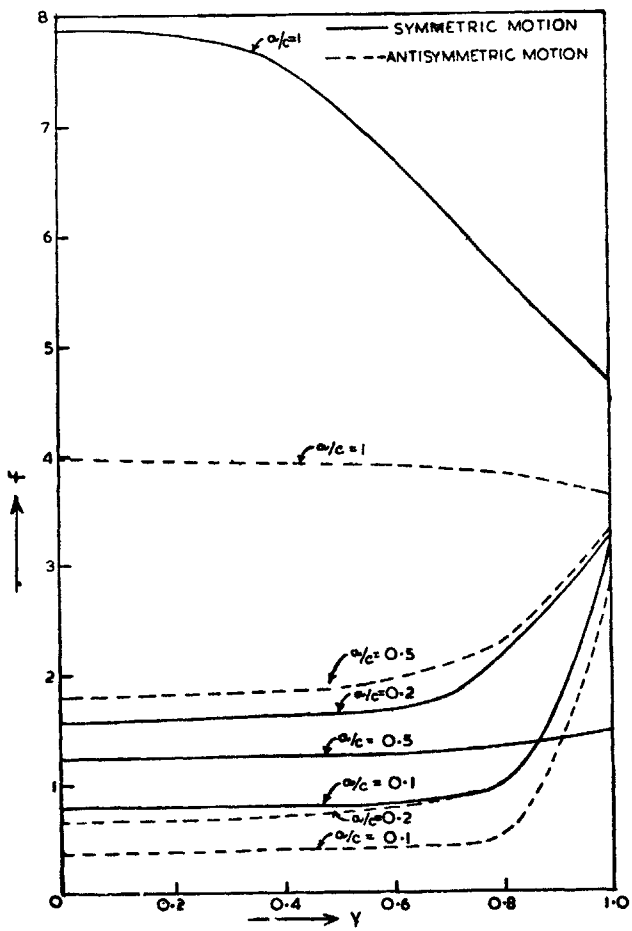

FIG. 1. Variation of frequency (f) with inner radius ratio (y).

decreases as inner radius ratio increases. From $y = 0.6$ onwards, the slope of the respective curves increases. The above conclusions are true for antisymmetric motion too. Further, the frequency is found to be less in antisymmetric motion when compared to its corresponding values of a/c (except for 0.5) for symmetric motion. At $y = 0.75$, the value of frequency for symmetric motion ($a/c = 0.1$) and antisymmetric motion ($a/c = 0.2$) are equal and from thereonwards respective frequency curves are coincident. In general, it is observed that frequency increases as diameter thickness ratio increases.

4. Torsional Vibrations of a Solid Composite Poroelastic Cylinder

The torsional vibrations of a solid composite poroelastic cylinder bonded end to end of different materials are considered. Let the interface of cylinder lies at $z = 0$ and other flat surfaces lie at $z = c$ and $z = -l$. The propagation mode shapes in this case can be written from eqn. (3.4) in the modified form as

$$u_{\theta j} = J_1(q_j r)\,[c_{1j} \cos(k_j z) + c_{2j} \sin(k_j z)] \exp(ipt), \; (j = 1, 2). \qquad \ldots(4.1)$$

$j = 1, 2$ correspond to different materials on two sides of the interface. The boundary condition to be satisfied on the curved surface is,

at $\quad r = a, \quad \sigma_{r\theta} = 0.$

This gives

$$J_2(q_j a) = 0, \quad j = 1, 2. \qquad \ldots(4.2)$$

Putting

$$q_1 = q_2 = q$$

eqn. (4.2) determines qa for different modes of vibration. The values of it are given in Abramowitz *et al.* (1965). In particular, the first four mode of vibration are

$$qa = 0, 5.136, 8.417, 11.62.$$

The boundary conditions at the flat ends are

at $\quad z = c, \quad (\sigma_{\theta z})_1 = 0$

$$z = -l, \; (\sigma_{\theta z})_2 = 0$$

$$z = 0, (u_\theta)_1 = (u_\theta)_2, (\sigma_{\theta z})_1 = (\sigma_{\theta z})_2. \qquad \ldots(4.3)$$

The pore-pressure (σ) satisfies identically. From eqns. (2.1), (4.1) and (4.3) four homogeneous equations in four unknowns are obtained. The requirement that the determinant of the coefficients must vanish gives the frequency equation to be

$$\frac{\tan(k_1 c)}{\tan(k_2 l)} = -\frac{N_2 k_2}{N_1 k_1}. \qquad \ldots(4.4)$$

From eqns. (3.3) and (4.4), one obtains

$$\frac{\tan\,[(s_1^2 - q^2)^{1/2}c]}{\tan\,[(s_2^2 - q^2)^{1/2}l]} = -\,\frac{N_2(s_2^2 - q^2)^{1/2}}{N_1(s_1^2 - q^2)^{1/2}}. \qquad \ldots(4.5)$$

The frequency eqn. (4.5) can be expressed in non-dimensional form as

$$\frac{\tan\left[(f^2m_4^{(1)} - q^2a^2)^{1/2}\cdot\left(\frac{c}{a}\right)\right]}{\tan\left[(f^2m_4^{(2)} - q^2a^2)^{1/2}\cdot\left(\frac{l}{a}\right)\right]} = -\,\frac{\beta^{(2)}(f^2m_4^{(2)} - q^2a^2)^{1/2}}{\beta^{(1)}(f^2m_4^{(1)} - q^2a^2)^{1/2}} \qquad \ldots(4.6)$$

where $\beta = N/H$, $H = P + 2Q + R$. The superscripts 1 and 2 in brackets correspond to the two different materials. When liquid effects are vanishingly small, the results of classical theory are obtained as a particular case.

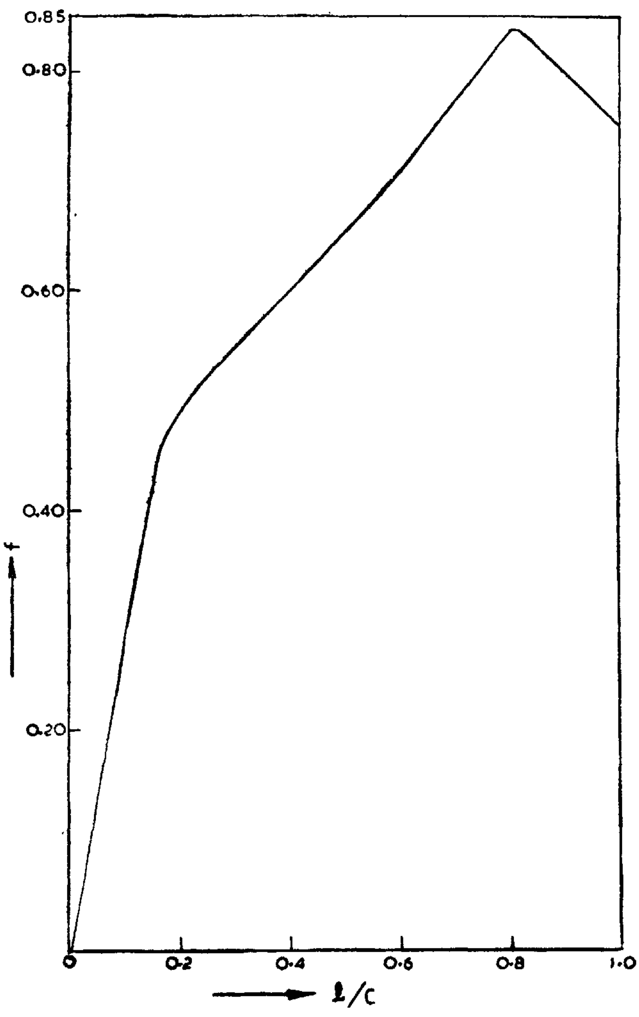

FIG. 2. Variation of frequency (f) with respect to ratio of lengths of cylinder (l/c).

The frequency eqn. (4.6) is an equation involving frequency (f), mode of vibration (qa), diameter-thickness ratio (a/c), material constant (m_4), shear modulus (β) and change in ratio of lengths of cylinder (l/c). The frequency (f) is computed from eqn. (4.6) corresponding to a given value of ratio of lengths of cylinder (l/c) taking the following data for the least mode of vibration, viz.,

$$m_1^{(1)} = 0.65,\ m_2^{(1)} = -0.15,\ m_3^{(1)} = 0.65,\ \beta^{(1)} = 0.21$$

$$m_1^{(2)} = 0.8, m_2^{(2)} = 0,\ m_3^{(2)} = 0.2, \beta^{(2)} = 0.25, a/c = 0.20.$$

The frequency versus ratio of lengths of cylinder is exhibited graphically in Fig. 2.

From the figure, it is noticed that frequency is zero at $l/c = 0$. In the interval (0, 0.8) of l/c, the frequency is found to increase whereas in the interval (0.8, 1) frequency decreases.

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