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WAVE PROPAGATION IN RANDOM MEDIA

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A dynamical equation for a field where the field characteristic $\epsilon(\mathbf{x})$ is a random function of position is discussed. Such equations occur in connexion with acoustic wave propagation in fluids. The equation governing the mean of the field quantity, $\{\varphi(\mathbf{x})\}$, is obtained for the infinite medium. The dispersion relations are obtained for waves propagating in the infinite medium.

INTRODUCTION

The theory of wave propagation in random media is essentially the study of linear partial differential equations with random functions as coefficients. Beran and McCoy (1970) considered the differential equation in the form

$$Lu = f \qquad \ldots(1)$$

where L is a random differential operator whose coefficients are random functions dependent on time and space coordinates, u is the field quantity and f is the random source term. It was shown that the mean field $\{u\}$ satisfies the equation

$$[\{L\} - \{L'[\{L\} + (I - P)L']^{-1} L'\}]\{u\}$$
$$= \{f\} - \{L'[\{L\} + (I - P) L']^{-1} f'\} \qquad \ldots(2)$$

where I is the identity operator and P is the averaging operator and

$$L = \{L\} + L', \quad u = \{u\} + u', \quad f = \{f\} + f'.$$

Here { } indicates the ensemble average of the relevant quantity and (′) denotes the fluctuating part of the indicated quantity about the ensemble average.

In the present paper, using the results of Beran and McCoy (1970), we have studied the random differential equation

$$\frac{\partial}{\partial x_j}\left[\epsilon(\mathbf{x})\frac{\partial\varphi(\mathbf{x})}{\partial x_j}\right] + \rho(\mathbf{x})\,\omega^2\varphi(\mathbf{x}) = F(\mathbf{x}) \qquad \ldots(3)$$

where $\epsilon(\mathbf{x})$ is assumed to be a statistically homogeneous random function of position, $F(\mathbf{x})$ is the source term taken to be non-random, $\rho(\mathbf{x})$ is the density of the medium taken to be non-random. We have considered the field equation (3) in an infinite space. Choosing the appropriate Green's function we have obtained the integro-differential equation governing the averaged field quantity $\{\phi(\mathbf{x})\}$. The frequency equation corresponding to wave propagation has been derived. Thereafter, following Beran and McCoy (1970), the small perturbation case for small wave number K is discussed and the equation governing the field quantity $\{\varphi(\mathbf{x})\}$ is obtained as a differential equation. This agrees with that obtained in Beran and McCoy (1970) in the limiting case when $K \to 0$ and our basic equation tends to eqn. (7) of Beran and McCoy (1970). The phase and group velocities have been obtained for large wave number by use of asymptotic results.

It may be pointed out that the acoustic wave equation in fluids is a particular case of the more general dynamical eqn. (3) discussed in this paper. The basic equations representing acoustic wave propagation in fluids, to first order in the small quantities p, $\mathbf{u}$, neglecting viscosity and heat conduction, are (Morse and Ingard 1968) :

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\text{grad}\, p + \mathbf{F}\rho, \qquad k \frac{\partial p}{\partial t} = -\text{div}\, \mathbf{u}$$

where ρ is density, p is fluid pressure, k is compressibility, $\mathbf{u}$ is the velocity vector and $\mathbf{F}$ is the body force per unit mass. Assuming k to be independent of t, these equations give

$$k \frac{\partial^2 p}{\partial t^2} = \text{div} \left(\frac{1}{\rho} \text{grad}\, p \right) - \text{div}\, \mathbf{F}.$$

Again assuming harmonic time dependence with period $2\pi/\omega$,

$$\frac{\partial}{\partial x_j} \left(\frac{1}{\rho} \frac{\partial p}{\partial x_j} \right) + k\omega^2 p = \text{div}\, \mathbf{F}. \qquad \ldots(3')$$

Here ρ and k can easily be subjected to random fluctuations. We have discussed the relatively simple case where only ρ is a random function and k is non-random.

An equation similar to (3) is obtained for heat conduction in random media for harmonic time-dependence but ω^2 is then replaced by an imaginary quantity.

The Problem

We proceed to consider eqn. (3) and write

$$\epsilon(\mathbf{x}) = \{\epsilon\} + \epsilon'$$

$$L = \frac{\partial}{\partial x_j} \left[\epsilon(\mathbf{x}) \frac{\partial}{\partial x_j} \right] + \rho\omega^2$$

$$\{L\} = \{\epsilon(\mathbf{x})\} \frac{\partial^2}{\partial x_j \partial x_j} + \rho\omega^2$$

$$L' = \frac{\partial}{\partial x_j}\left[\epsilon'(\mathbf{x})\frac{\partial}{\partial x_j}\right].$$

Let $\mathbf{x}$ locate the field point and $\mathbf{x}'$ locate the source point in the field under consideration. Then $G(\mathbf{x}, \mathbf{x}')$ will denote the appropriate Green's function required for obtaining the inverse operator $\{L\}^{-1}$, involved in eqn. (3).

Therefore

$$\{L\}^{-1} u(\mathbf{x}) = \left[\{\epsilon(\mathbf{x})\}\frac{\partial^2}{\partial x_j\, \partial x_j} + \rho\omega^2\right]^{-1} u(\mathbf{x})$$

$$= \frac{1}{\{\epsilon\}} \int G(\mathbf{x}, \mathbf{x}',)\, u(\mathbf{x}')\, d\mathbf{x}'$$

where the integral is to be taken over the entire volume, and $u(\mathbf{x})$ represents a generic function of position. As L' has the same form as in the paper of Beran and McCoy (1970), their results may be used in this case also with the appropriate change of Green's function. Thus $\{\varphi(\mathbf{x})\}$ can be shown to satisfy the equation

$$\{\epsilon(\mathbf{x})\}\frac{\partial}{\partial x_j}\{E_j(\mathbf{x})\} + \rho\omega^2\{\varphi(\mathbf{x})\} - \frac{\partial}{\partial x_j}[\mathcal{K}_{ij}(\mathbf{x}, \mathbf{x}')\{E_j(\mathbf{x}')\}] = F(\mathbf{x}) \quad \ldots(4)$$

where $\mathcal{K}_{ij}$ has meaning similar to that given in Beran and McCoy (1970) and

$$E_j(\mathbf{x}) = \frac{\partial}{\partial x_j}\{\varphi(\mathbf{x})\}.$$

Infinite Space

Let us consider eqn. (3) in an infinite space. We assume $F(\mathbf{x})$ to be a local source term confined to some finite volume while $\varphi(\mathbf{x})$ vanishes at infinity. We also choose the statistics of the $\epsilon(\mathbf{x})$ field to be homogeneous and isotropic. In this space, the free space Green's function is $G(\mathbf{x}, \mathbf{x}')$ which is a function of $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, where $\mathbf{x}$ is the field point and $\mathbf{x}'$ is the source point.

Then

$$G_j(\mathbf{x}, \mathbf{x}') = \frac{\partial G}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{4\pi}\frac{\exp(iK_0 r}{r}\right)$$

$$= \frac{-1}{4\pi} r_j \exp(iK_0 r)\left(\frac{1}{r^3} - \frac{iK_0}{r^2}\right) \quad \ldots(5)$$

where $\quad i = \sqrt{-1}, \quad K_0 = \sqrt{\rho/\{\epsilon\}}\,.\,\omega \quad \ldots(6)$

and $\quad \mathbf{x}_j - \mathbf{x}'_j = \mathbf{r}_j, \quad |\mathbf{r}_j| = r_j.$

Then the expression for $\alpha_{jq}(\mathbf{x}_n, \mathbf{x}')$ of eqn. (31) of Beran and McCoy (1970) becomes in our case

$$\alpha_{jq}(\mathbf{x}_n, \mathbf{x}') = \left[(1 - iK_0 r)\, \delta_{jq} + r_j r_q \left(K_0^2 + \frac{3iK_0}{r} - \frac{3}{r^2} \right) \right] \exp(iK_0 r) \qquad \ldots(7)$$

and eqn. (4) reduces to, under the same assumptions as were necessary for deriving eqn. (44) of Beran and McCoy (1970):

$$\epsilon^* \nabla^2 \{\varphi(\mathbf{x})\} + \frac{\partial}{\partial x_i} \int M_{ij}(\mathbf{x} - \mathbf{x}') \frac{\partial}{\partial x_i} \{\varphi(\mathbf{x}')\}\, d\mathbf{x}' + \rho\omega^2 \{\varphi(\mathbf{x})\} = F(\mathbf{x}) \qquad \ldots(8)$$

where $$\epsilon^* = \{\epsilon\} - \frac{1}{3} \frac{C\Sigma}{\{\epsilon\}} - \eta \qquad \ldots(9)$$

and $$M_{ij}(\mathbf{x} - \mathbf{x}') = G_{ij}(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} - \mathbf{x}') \int G_{ij}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'. \qquad \ldots(10)$$

Taking Fourier transform of both sides of (8) we obtain

$$-\epsilon^* K^2 \{\hat{\varphi}(\mathbf{K})\} - K_i \hat{M}_{ij}(\mathbf{K})\, K_j \{\hat{\varphi}(\mathbf{K})\} + \rho\omega^2 \{\hat{\varphi}(\mathbf{K})\} = \hat{F}(\mathbf{K}).$$

Writing $K^2 \hat{M}(\mathbf{K}) = K_i K_j \hat{M}_{ij}(\mathbf{K})$, we have

$$K^2 [-\epsilon^* - \hat{M}(\mathbf{K})] \{\hat{\varphi}(\mathbf{K})\} + \rho\omega^2 \{\hat{\varphi}(\mathbf{K})\} = \hat{F}(\mathbf{K})$$

or $$\{\hat{\varphi}(\mathbf{K})\} = \frac{\hat{F}(\mathbf{K})}{\rho\omega^2 - K^2 [\epsilon^* + \hat{M}(\mathbf{K})]}. \qquad \ldots(11)$$

Now taking Fourier's inversion on both sides of eqn. (11), the mean field $\{\varphi(\mathbf{x})\}$ is obtained in the form:

$$\{\varphi(\mathbf{x})\} = \frac{1}{(2\pi)^3} \iiint \iiint \frac{F(\mathbf{x}') \exp(-i\mathbf{K}.\mathbf{r})\, d\mathbf{K}\, d\mathbf{x}'}{\rho\omega^2 - \mathbf{K}^2 [\epsilon^* + \hat{M}(|K|)]} \qquad \ldots(12)$$

where $\mathbf{x} - \mathbf{x}' = \mathbf{r}$.

If $F(\mathbf{x}')$ represents a localised source i.e. $F(\mathbf{x}') = 0$ outside a certain limited region, the K-integral

$$\iiint \frac{\exp(-i\mathbf{K}\mathbf{r})\, d\mathbf{K}}{\rho\omega^2 - K^2 [\epsilon^* + \hat{M}(|K|)]} = \frac{2\pi}{ir} \int_{-\infty}^{\infty} \frac{\exp(iKr)\, K dK}{\rho\omega^2 - K^2 [\epsilon^* + \hat{M}(|K|)]}$$

can be evaluated by the help of the theory of asymptotic expansion of Fourier transforms of generalized functions (Lighthill 1958; also see Appendix) for large $r = |\mathbf{x} - \mathbf{x}'|$.

Assuming $$\rho\omega^2 - K^2 [\epsilon^* + \hat{M}(|K|)] = 0 \qquad \ldots(13)$$

to have simple poles at $K = \pm K_1, \pm K_2, \ldots, \pm K_n$, ($K_j > 0$, $j = 1, 2, \ldots, n$), it can be shown by using results on asymptotic expansions due to Lighthill (1958) and following the procedure of Mitra (1958) that

$$\{\varphi(\mathbf{x})\} = \frac{2}{\pi}\sum_{j=1}^{n} \iiint \frac{F(\mathbf{x}')}{r} \cdot \frac{\cos(rK_j)}{\hat{M}'(|K_j|)}\, d\mathbf{x}' \qquad \ldots(14)$$

where $\hat{M}'(K)$ stands for derivative of $\hat{M}(K)$ with respect to K.

Hence the principal contributions to wave propagation at large distances are obtained from zeros of (13).

Thus groups of waves are propagated with velocities determined by

$$\rho\omega^2 = K^2 [\epsilon^* + \hat{M}(|K|)]$$

or

$$\rho c^2 = \epsilon^* + \hat{M}(|K|). \qquad \ldots(15)$$

Equation (15) may therefore be regarded as the dispersion relation for the propagation of waves. It may be noted that for a non-random medium, $\hat{M}(|K|) = 0$, $\epsilon^* = \{\epsilon\}$, so that $K = K_0$.

The Case of Small Perturbation

For the small perturbation case (Beran and McCoy 1970, p. 253), it can be shown that

$$K^2\hat{M}(\mathbf{K}) = K_iK_j\hat{M}_{ij}(\mathbf{K})$$

$$= \frac{-K_iK_j}{4\pi\{\epsilon\}} \int \frac{1}{r^3}\, \alpha_{ij}(r)\, c(r) \exp(i\mathbf{K}\cdot\mathbf{r})\, d\mathbf{r}$$

$$= \frac{-1}{4\pi\{\epsilon\}} \int \frac{c(r)}{r^3} \exp(iK_0r)\, [(1 - iK_0r)\, \delta_{ij}K_iK_j$$

$$+ \left(K_0^2 + \frac{3iK_0}{r} - \frac{3}{r^2}\right) K_iK_jr_ir_j] \exp(i\mathbf{K}.\mathbf{r})\, d\mathbf{r}.$$

Taking $\mathbf{K}$ along Z-axis we have

$$\mathbf{K}_i \cdot \mathbf{r}_i = Kr\cos\theta = \mathbf{K}_j \cdot \mathbf{r}_j \qquad \ldots(16)$$

$$\therefore \quad \hat{M}(\mathbf{K}) = \frac{-1}{2\{\epsilon\}} \int_0^{\infty}\int_0^{\pi} \frac{c(r)}{r} \exp(iK_0r) \left[1 - iK_0r + \left(K_0^2 + \frac{3iK_0}{r} - \frac{3}{r^2}\right) r^2\cos^2\theta\right]$$

$$\times \exp(iKr\cos\theta)\sin\theta\, dr\, d\theta. \qquad \ldots(17)$$

Expanding exp (iKr cos θ) in powers of K and neglecting higher powers, we get, for small k,

$$\hat{M}(K) = a_0(K_0) + a_1(K_0)K^2 + a_2(K_0)\, K^4 + O\,(K^6) \quad \ldots(18)$$

where

$$\left.\begin{aligned} a_0 &= \frac{-K_0^2}{3\{\epsilon\}} \int_0^\infty c(r) \exp(iK_0 r)\, r dr \\ a_1 &= \frac{1}{30\{\epsilon\}} \int_0^\infty c(r) \exp(iK_0 r) \cdot r[3K_0^2\, r^2 + 4iK_0 r - 4]\, dr \\ a_2 &= \frac{-1}{840\{\epsilon\}} \int_0^\infty c(r)\, r^3 \exp(iK_0 r)[5K_0^2\, r^2 + 8iK_0 r - 8]\, dr. \end{aligned}\right\} \quad \ldots(19)$$

We have assumed here that $c(r)$ decays with sufficient rapidity as r increases so that $\int_0^\infty c(r)\, r^7\, dr$ is bounded.

Substituting in (11),

$$\{\hat{\varphi}(\mathbf{K})\} = \frac{\hat{F}(\mathbf{K})}{\rho\omega^2 - K^2[\epsilon^* + a_0 + a_1 K^2 + a_2 K^4]}. \quad \ldots(20)$$

Taking inverse, we have

$$\{\varphi(\mathbf{x})\} = \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{\hat{F}(\mathbf{K}) \exp(-i\mathbf{K}.\mathbf{x})\, d\mathbf{K}}{\rho\omega^2 - K^2[\epsilon^* + a_0 + a_1 K^2 + a_2 K^4]}. \quad \ldots(21)$$

From (20) we find

$$(\epsilon^* + a_0)\, \nabla^2\{\varphi(\mathbf{x})\} - a_1\nabla^4\{\varphi(\mathbf{x})\} + a_2\nabla^6\{\varphi(\mathbf{x})\} + \rho\omega^2\{\varphi(\mathbf{x})\} = F(\mathbf{x}). \quad \ldots(22)$$

This is the differential equation governing $\{\varphi(\mathbf{x})\}$.

The frequency eqn. (13) becomes, in the case of small wavenumber

$$\rho c^2 = \epsilon^* + a_0 + a_1 K^2 + a_2 K^4. \quad \ldots(23)$$

The group velocity U is given by $U = \dfrac{d\omega}{dK}$, $\omega = cK$. It is to be remembered here that a_0, a_1, a_2 depend on K_0 and

$$K_0 = \omega\sqrt{\rho/\{\epsilon\}} = cK\sqrt{\rho/\{\epsilon\}}. \quad \ldots(24)$$

As $K_0 \to 0$, we observe that,

$$\left.\begin{aligned} a_0(K_0) &\to 0 \\ a_1(K_0) &\to \frac{-2}{15\{\epsilon\}} \int_0^\infty rc(r)\,dr \\ a_2(K_0) &\to \frac{1}{105\{\epsilon\}} \int_0^\infty r^3 c(r)\,dr \end{aligned}\right\} . \qquad \ldots(25)$$

As $K_0 \to 0$, our basic eqn. (3) tends to eqn. (7) of Beran and McCoy (1970). These results agree with the coefficients obtained by expanding $\hat{M}(K)$ of their paper in powers of K.

Also setting $K_0 r = x$, we find that as $K_0 \to \infty$,

$$a_1(K_0) = \frac{1}{30\{\epsilon\}} \cdot \frac{1}{K_0^2} \int_0^\infty c\left(\frac{x}{K_0}\right) e^{ix}\,[3x^2 + 4ix - 4]\, x dx \to 0,$$

and $a_2(K_0) \to 0$, assuming the same behaviour of $c(r)$ as before.

Phase and Group Velocities for Large K:

Equation (17) may be written as

$$\hat{M}(K) = \frac{-1}{\{\epsilon\}} \int_0^\infty \frac{c(r)}{r} \exp\,(iK_0 r)\Big[(1 - iK_0 r)\frac{\sin Kr}{Kr} - (K_0^2 r^2 + 3iK_0 r - 3)\frac{d^2}{d(Kr)^2}\left(\frac{\sin Kr}{Kr}\right)\Big] dr. \qquad \ldots(26)$$

We now assume that $c(r)$, which is independent of K_0, satisfies the following conditions: (i) $c(r) \to 0$ as $r \to \infty$; (ii) $rc'(r) \to 0$ as $r \to \infty$; (iii) $\frac{c(r)}{r} = c_0 + c_1 r + c_2 r^2 + \ldots$, for small r.

Then integrating by parts and making use of the result

$$\underset{\lambda \to \infty}{\mathrm{Lt}} \int_0^\infty f(x) \frac{\sin \lambda x}{x}\, dx = \frac{\pi}{2} f(+0),$$

we obtain

$$\int_0^\infty \frac{c(r)}{r} \exp\,(iK_0 r)\,(1 - iK_0 r)\,\frac{\sin Kr}{Kr}\, dr \approx \frac{\pi c_0}{2K} \text{ as } K \to \infty,$$

and

$$- \int_0^\infty \frac{c(r)}{r} \exp(iK_0 r)(K_0^2 r^2 + 3iK_0 r - 3) \frac{d^2}{dr^2}\left(\frac{\sin Kr}{Kr}\right) dr$$

$$\approx 3c_1 + \frac{6c_2 + c_0 K_0^2}{K} \cdot \frac{\pi}{2} \text{ as } K \to \infty. \qquad \ldots(26')$$

Hence

$$\hat{M}(K) \approx \frac{-1}{\{\epsilon\}} \left[\frac{\pi c_0}{2K} + \frac{3c_1}{K^2} + \frac{\pi}{2K^3}(6c_2 + c_0 K_0^2) \right] \text{as } K \to \infty.$$

Since $c(r)$ is independent of K_0, hence c_0, c_1, c_2 are also independent of K_0, i.e. ω. Hence it can be shown from (15) that the phase velocity c is given by

$$c = \sqrt{\frac{\epsilon^*}{\rho}} \cdot \left[1 + A_1 \cdot \frac{1}{K} + A_2 \cdot \frac{1}{K^2} + A_3 \cdot \frac{1}{K^3} + O\left(\frac{1}{K^4}\right)\right] \qquad \ldots(27)$$

where

$$A_1 = \frac{\pi c_0}{2\epsilon^*\{\epsilon\}}\left(1 + \frac{\epsilon^*}{\{\epsilon\}}\right) \qquad \ldots(27')$$

$$A_2 = \frac{1}{\epsilon^*\{\epsilon\}}\left(3c_1 - \frac{\pi^2 c_0^2}{4\{\epsilon\}^2} - \frac{\pi^2 c_0^2 \epsilon^*}{4\{\epsilon\}^3} + \frac{\pi^2 c_0^2}{4\epsilon^*\{\epsilon\}}\left(1 + \frac{\epsilon^*}{\{\epsilon\}}\right)^2\right)$$

$$A_3 = \frac{\pi}{\epsilon^*\{\epsilon\}}\left(3c_2 - \frac{3c_0 c_1}{2\{\epsilon\}^2} + \frac{\pi^2 c_0^3}{8\{\epsilon\}^4} + \frac{\pi^2 c_0^3 \epsilon^*}{8\{\epsilon\}^5}\right.$$

$$+ \frac{c_0}{\epsilon^*\{\epsilon\}}\left(1 + \frac{\epsilon^*}{\{\epsilon\}}\right)\left(3c_1 - \frac{\pi^2 c_0^2}{4\{\epsilon\}^2} - \frac{\pi^2 c_0^2 \epsilon^*}{4\{\epsilon\}^3}\right)$$

$$\left. + \frac{\pi^2 c_0^3}{8\epsilon^{*2}\{\epsilon\}^2}\left(1 + \frac{\epsilon^*}{\{\epsilon\}}\right)^3\right).$$

Hence the group velocity $U = \dfrac{d\omega}{dK} = \dfrac{d(cK)}{dK}$ is given by

$$U = \sqrt{\frac{\epsilon^*}{\rho}} \cdot \left[1 - A_2 \cdot \frac{1}{K^2} - 2A_3 \cdot \frac{1}{K^3} - O\left(\frac{1}{K^4}\right)\right]. \qquad \ldots(28)$$

Therefore phase and group velocities both tend to the limiting value

$\sqrt{\epsilon^*/\rho}$ as $K \to \infty$.

These results, as we have stated earlier, apply to acoustic waves in random media.

Application to Acoustic Waves

The propagation of acoustic waves is governed by eqn. (3′) in which we assume that only ρ admits of random fluctuation, k being non-random, so that we take

$$\{\epsilon\} = \left\{\frac{1}{\rho}\right\}. \qquad \ldots(29)$$

Also K_0 in this case is given by

$$K_0 = \omega \sqrt{k/\{\epsilon\}} = cK \sqrt{k/\{\epsilon\}}. \qquad \ldots(30)$$

Then following eqn. (23), we obtain an equation for small wavenumbers, given by

$$c^2 \approx \frac{\epsilon^*}{k} \cdot \left[\frac{1 - AK^2/\epsilon^*}{1 + 5AK^2/2\{\epsilon\}} \right] \qquad \ldots(31)$$

where $$A = \frac{2}{15\{\epsilon\}} \int_0^\infty rc(r)\, dr.$$

Equation (31) expresses the dispersion relation for acoustic wavesfor small wavenumbers. From (31) it is clear that if A is assumed positive, the phase velocity increases or decreases according as K decreases or increases. If however, A is assumed negative, the phase velocity increases or decreases according as K increases or decreases.

Again for large wavenumbers, the phase velocity c of acoustic waves is given by

$$c \approx \sqrt{\frac{\epsilon^*}{k}} \cdot \left[1 + \frac{A_1}{K} + O\left(\frac{1}{K^2} \right)\right] \qquad \ldots(32)$$

where ϵ^* is the effective constant of the random media defined for constant average fields given by (See Beran (1968) Chap. 5) :

$$\epsilon^* = \{\epsilon\} - \tfrac{1}{3}\, c(0)/\{\epsilon\}.$$

Also A_1 is given by (27′) with these values of ϵ^* and $\{\epsilon\}$.

Now for both high and low wavenumbers we observe that if the effect of correlation coefficient $c(r)$ is small, then

$$c \approx \sqrt{\frac{\epsilon^*}{k}}. \qquad \ldots(33)$$

One effect of random fluctuation of ρ for both small and large wavenumbers, is thus to replace ϵ in the expression $\sqrt{\epsilon/\rho}$ for the phase velocity by

$$\epsilon^* = \{\epsilon\} - \frac{1}{3} \frac{c(0)}{\{\epsilon\}}.$$

This is greater or less than the average value of $\{\epsilon\}$ according as the correlation coefficient $c(r)$ at $r = 0$ is positive or negative.

The second effect is to introduce dispersion into the process of wave propagation, as is shown for both small and large wavenumbers by eqns. (31) and (32). In both these cases, the phase velocity approaches the limiting value $\sqrt{\epsilon^*/k}$. For a positive correlation coefficient $c(r)$, it is easily seen that A, $A_1 > 0$, so that the phase velocity in the random medium is less than the limiting value $\sqrt{\epsilon^*/k}$ for small wavenumbers, while it is greater than this value for large wavenumbers. The reverse is the case if the correlation coefficient $c(r)$ is negative.

Actually the sufficient conditions for these conclusions to hold are $A, A_1 \gtrsim 0$, but $c(r) \gtrsim 0$ is more easily interpreted in physical terms.

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APPENDIX

The

$$K\text{-integral} = \iiint \frac{\exp(-i\mathbf{K}.\,\mathbf{r}\,d\mathbf{K})}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(\,|\,K\,|\,)]}$$

$$= 2\pi \int_0^\infty \int_0^\pi \frac{K^2 \exp(-iKr\cos\theta)\sin\theta\, d\theta\, dK}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(\,|\,K\,|\,)]}$$

$$= \frac{2\pi}{ir} \int_0^\infty \frac{(\exp(iKr) - \exp(-iKr))\, KdK}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(\,|\,K\,|\,)]}.$$

Now putting $-K$ for K in the second integral, it can be shown that the K-integral

$$= \frac{2\pi}{ir} \int_{-\infty}^{\infty} \frac{\exp(iKr)\, KdK}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(\,|\,K\,|\,)]}.$$

Next we take

$$\frac{K}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(\,|\,K\,|\,)]} = \frac{\Phi_j(K)}{K^2 - K_j^2} = \frac{\Phi_{j_1}(K)}{K - K_j} = \frac{\Phi_{j_2}(K)}{K + K_j}.$$

where $\Phi_{j1}(K)$, $\Phi_{j2}(K)$ are holomorphic in the neighbourhood of $+K_j$, $-K_j$ respectively. Let us define $F_{j1}(K)$, $F_{j2}(K)$ such that

$$F_{j1}(K) = \frac{\Phi_{j1}(K_j)}{K - K_j} + \Phi'_{j1}(K_j) + \frac{K - K_j}{2!}\Phi''_{j1}(K_j)$$

$$F_{j2}(K) = \frac{\Phi_{j2}(-K_j)}{K + K_j} + \Phi'_{j2}(-K_j) + \frac{K + K_j}{2!}\Phi''_{j2}(-K_j)$$

where the derivatives are assumed to exist.

Now $$\frac{K}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(|K|)]} - F_{j1}(K) \text{ and } \frac{K}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(|K|)]} - F_{j2}(K)$$

have absolutely integrable second derivatives in the intervals including $+K_j$ and $-K_j$ respectively.

The Fourier transform of $\dfrac{K}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(|K|)]}$

= sum of Fourier transforms for all j, of $F_{j1}(K)$ and $F_{j2}(K) + O(r^{-2})$, $r = |\mathbf{x} - \mathbf{x}'|$. (See Lighthill 1958, p. 52).

But Fourier transform of $F_{j1}(K) = \pi i \exp(-irK_j)\,\Phi_{j1}(K_j)$ + sum of derivatives of multiples of $\delta\left(\dfrac{r}{2\pi}\right)$ and Fourier transform of $F_{j2}(K) = -\pi i \exp(irK_j)\,\Phi_{j2}(-K_j)$ + sum of derivatives of multiples of $\delta\left(\dfrac{r}{2\pi}\right)$.

Also $$\Phi_{j1}(K) = \frac{K(K - K_j)}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(|K|)]}.$$

Making $K \to K_j$, we obtain $\Phi_{j1}(K_j) = \dfrac{K_j}{-\hat{M}'(|K_j|)}$.

Similarly $$\Phi_{j2}(-K_j) = \frac{K_j}{-\hat{M}'(|K_j|)}.$$

Thus for large $r = |\mathbf{x} - \mathbf{x}'|$

$$\frac{2\pi}{ir}\int_{-\infty}^{\infty} \frac{\exp(iKr)\,KdK}{\rho\omega^2 - K^2[\epsilon^* + \hat{M}(|K|)]} \approx 2\pi^2 \sum_j \frac{1}{r}\left(\frac{\exp(-irK_j)}{\hat{M}'(|K_j|)} + \frac{\exp(irK_j)}{\hat{M}'(|K_j|}\right)$$

Therefore $$\{\Phi(\mathbf{x})\} \approx \frac{2}{\pi}\sum_j \iiint \frac{F(\mathbf{x}')}{r}\,K_j\,\frac{\cos(rK_j)}{\hat{M}'(|K_j|)}\,d\mathbf{x}'.$$