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FOX'S *H*-FUNCTION AND ELECTRIC CIRCUIT THEORY

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The aim of this paper is to obtain the value of the charge at any time t in a simple electric circuit consisting of resistance, inductance, capacitance and a source of electromotive force $E_0P(t)$, when $P(t)$ is taken in terms of the Fox's H-functions. This function is believed to be quite general in nature because it includes a number of well-known elementary functions as its particular cases. Evidently, therefore, our results would apply to a wide variety of useful functions (or products of several such functions) occurring frequently in mathematical physics and engineering.

1. A USEFUL INTEGRAL

The following integral which is a special case of our recent result (Goyal and Agrawal 1981) will be required in the sequel

$$\int_0^t x^{\rho-1}(c+bx)^{-\lambda} e^{RX/2L} \sin\{\omega(t-x)\}\, H_{P,Q}^{M,N}\left[zx^u(c+bx)^{-\epsilon} \middle| \begin{matrix} (a_j,A_j)_{1,P} \\ (b_j,B_j)_{1,Q} \end{matrix} \right] dx$$

$$= t^{\rho+1} c^{-\lambda} \sum_{r,k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m} (-b/c)^m (Rt/2L)^r}{r!m!}$$

$$\times H_{P+2,Q+2}^{M,N+2}\left[zt^u c^{-\epsilon} \middle| \begin{matrix} (1-\lambda-m,\epsilon),\ (1-\rho-r-m,u),\ (a_j,A_j)_{1,P} \\ (b_j,B_j)_{1,Q},\ (1-\lambda,\epsilon),\ (-1-2k-\rho-r-m,u) \end{matrix} \right] \quad \ldots(1.1)$$

$$= t^{\rho+1} c^{-\lambda} \sum_{r,k=0}^{\infty} \sum_{m=0}^{r} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+r} (b/c)^m (R/2L)^{r-m} (-)_m r)}{r!m!}$$

$$\times H_{P+2,Q+2}^{M,N+2}\left[zt^u c^{-\epsilon} \middle| \begin{matrix} (1-\lambda-m,\epsilon),\ (1-\rho-r,u)\ (a_j,A_j)_{1,P} \\ (b_j,B_j)_{1,Q},\ (1-\lambda,\epsilon),\ (-1-2k-r-\rho,u) \end{matrix} \right] \quad \ldots(1.2)$$

$$= t^{\rho+1} c^{-\lambda} e^{Rt/2L} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} \sum_{m=0}^{r-2k} \frac{(-1)^k (\omega)^{2k+1} (t)^r (-b/c)^m \Gamma(2+r-m)}{\Gamma(2+2k)(r-2k-m)!\, m!} \times$$

(*equation continued on p.* 40)

$$\times H^{M,N+2}_{P+2,Q+2}\left[zt^u c^{-\epsilon} \left| \begin{matrix} (1-\lambda-m,\epsilon),\ (1-\rho-m,u),\ (a_j, A_j)_{1,P} \\ (b_j,B_j)_{1,Q},\ (1-\lambda,\epsilon),\ (-1-\rho-r,u) \end{matrix} \right. \right]$$

$$(-R/2L)^{r-2k-m} \qquad \ldots(1.3)$$

Throughout this paper $H^{M,N}_{P,Q}\left[z \left| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right. \right]$ stands for the well-known Fox's H-function. An interesting and useful account of this function can be found, for example, in Mathai and Saxena (1978) and Srivastava (1980). Here $(a_j, A_j)_{1,P}$ abbreviates the parameter sequence $(a_1, A_1), \ldots, (a_P, A_P)$, and so on. Also, the symbol $[r/2]$ stands for the greatest integer in $r/2$.

The conditions of validity of the integral are : Re $(\lambda) > 0$, min $(u, \epsilon) > 0$, $t > 0$, $| bt/c | < 1$, $A > 0$, $| \arg z | < (1/2) A\pi$, Re $(\rho) + u \min_{1 \leq j \leq M} \{\text{Re}\,(b_j/B_j)\} > 0$ and the series on the right-hand side converges absolutely, it being understood that

$$A = \sum_{j=1}^{N} A_j - \sum_{j=N+1}^{P} A_j + \sum_{j=1}^{M} B_j - \sum_{j=M+1}^{Q} B_j. \qquad \ldots(1.4)$$

Proof of (1.2) : To prove (1.2), we make use of known results [Mathai and Saxena 1978, p. 2, eqn. (1.1.1); Rainville 1971, p. 58, eqn. (3)] in (1.1).

Proof of (1.3) : On using the contour representation for Fox's H-function occurring on the right-hand side of (1.1), changing the order of integration and summation with respect to r, we find that R.H.S. of (1.1)

$$= t^{\rho+1} c^{-\lambda} \sum_{k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m} (-b/c)^m}{m!}$$

$$\times (1/2\pi i) \int_L \prod_{j=1}^{M} \Gamma(b_j - B_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + A_j s)$$

$$\times \left[\prod_{j=M+1}^{Q} \Gamma(1 - b_j + B_j s) \prod_{j=N+1}^{P} \Gamma(a_j - A_j s) \right]^{-1}$$

$$\times (zt^u c^{-\epsilon})^s \frac{\Gamma(\lambda + m + \epsilon s)\, \Gamma(\rho + m + us)}{\Gamma(\lambda + \epsilon s)\, \Gamma(2 + 2k + \rho + m + us)}$$

$$\times {}_1F_1 [\rho + m + us;\ 2 + 2k + \rho + m + us;\ Rt/2L]\, ds \qquad \ldots(1.5)$$

Now using Kummer's first formula [Rainville 1971, p. 125, eqn. (2)] in (1.5) and then applying series representation for ${}_1F_1$ thus obtained, we easily get

$$\text{R.H.S. of (1.1)} = t^{\rho+1} c^{-\lambda} e^{Rt/2L} \sum_{r,k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k+1} (t)^{2k+m}}{r! \; m!}$$
$$\times (-b/c)^m (-Rt/2L)^r (2+2k)_r$$
$$\times H^{M,N+2}_{P+2,Q+2}\left[zt^u c^{-\epsilon} \middle| \begin{matrix} (1-\lambda-m, \epsilon), (1-\rho-m, u), (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q}, (1-\lambda, \epsilon), (-1-2k-\rho-r-m, u) \end{matrix} \right] \quad \ldots(1.6)$$

Finally, applying to the results [Rainville 1971, p. 56, eqn. (1); p. 57, eqn. (7)] in (1.6), we arrive at (1.3).

2. Main Problem

If we consider an electric circuit consisting of resistance R, an inductance L, a condenser of capacity C and a source of electromotive force $E_0 P(t)$, where E_0 is constant and $P(t)$ is known function of time t, the charge $q(t)$ on the plates of condenser at any time t, satisfies the following second order differential equation

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 P(t). \quad \ldots(2.1)$$

The solution of this differential equation subject to initial conditions $q = Q$, $i = dq/dt = I$ when $t = 0$, is the standard result (Sneddon 1957, p. 95) and is given below

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-Rt/2L} \int_0^t P(\eta) e^{R\eta/2L} \sin\{\omega(t-\eta)\} d\eta \quad \ldots(2.2)$$

where, for convenience,

$$J(t) = e^{-Rt/2L} [Q \cos \omega t + (I_1/\omega) \sin \omega t] \quad \ldots(2.3)$$

$$I_1 = I + RQ/2L \text{ and } (1/LC) - (R^2/4L^2) = \omega^2 > 0. \quad \ldots(2.4)$$

Now we turn to the problem of finding out the charge $q(t)$ when $P(t)$ is taken in terms of the H-function. We shall discuss two useful cases.

(i) *Solution of* (2.1) *When $P(t)$ is Taken in Terms of the H-function*

Let

$$P(t) = t^{\rho-1} (c+bt)^{-\lambda} H^{M,N}_{P,Q}\left[zt^u (c+bt)^{-\epsilon} \middle| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right]. \quad \ldots(2.5)$$

Putting the above value of $P(t)$ in (2.2) and evaluating the η-integral with the help of (1.1) { or (1.2) or (1.3)}, we find that the value of the charge $q(t)$ is given by

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-Rt/2L} F_i(r, k, m, t), \quad i = 1, 2, 3 \quad \ldots(2.6)$$

where $J(t)$, $F_i(r, k, m, t)$ $(i = 1, 2, 3)$ stand for the quantities as given by (2.3), (1.1), (1.2) and (1.3) respectively and the conditions mentioned after (1.3) are satisfied.

It is interesting to note that the value of the current dq/dt can also be obtained from (2.6), by differentiating the series on its right-hand side term by term with respect to t. The process of term by term differentiation is assumed to be justified as the H-function being analytic function (Mathai and Saxena 1978, p. 3) and the resulting series of H-functions obtained in this case will be uniform convergent in any arbitrary domain $0 \leqq t \leqq a$.

A special case of the solution (2.6), which is of practical interest, follows easily by putting $R = 0$; thus we arrive at the following solution

$$q(t) = Q \cos \omega t + (I/\omega) \sin \omega t$$
$$+ c^{-\lambda} \sum_{k,m=0}^{\infty} \frac{(-1)^k (\omega)^{2k} (t)^{2k+m} (-b/c)^m E_0 t^{\rho+1}}{m!\ L}$$
$$\times H^{M,N+2}_{P+2,Q+2} \left[z t^u c^{-\epsilon} \left| \begin{matrix} (1-\lambda-m, \epsilon), (1-\rho-m, u), (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q}, (1-\lambda, \epsilon), (-1-2k-\rho-m, u) \end{matrix} \right. \right]. \quad \ldots(2.7)$$

If we take all A's and B's equal to one in (2.7) and apply the well-known Gamma multiplication formula, we get the solution of (2.1) given by Gupta and Goyal [1973, p. 737, eqn. (2.2)].

(ii) *Solution of* (2.1), *When $P(t)$ is a Series of Sines*

If we substitute $\eta = \sin \theta$ in (2.2), the value of the charge $q(t)$ is easily seen to be given by the following equivalent form

$$q(t) = J(t) + \frac{E_0}{\omega L} e^{-Rt/2L} \int_0^{\sin^{-1} t} P(\sin \theta)\, e^{R \sin \theta/2L} \sin\{\omega(t - \sin \theta)\} \cos \theta \, d\theta. \quad \ldots(2.8)$$

Now on taking

$$P(\sin t) = \sum_{r=0}^{\infty} H^{M+1,N+1}_{P+2,Q+2} \left[z \left| \begin{matrix} (1-\rho-r, h), (a_j, A_j)_{1,P}, (2-\rho+r, h) \\ (\tfrac{3}{2}-\rho, h), (b_j, B_j)_{1,Q}, (1-\rho, h) \end{matrix} \right. \right]$$

$$\sin\ (2r+1)t = (\sqrt{\pi}/2)(\sin t)^{1-2\rho} H^{M,N}_{P,Q} \left[z(\sin t)^{-2h} \left| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right. \right] \quad \ldots(2.9)$$

[by virtue of a known result (Bajpai 1969, p. 705)] in (2.8) and again replacing $\sin\theta$ by x, we find that

$$q(t) = J(t) + \frac{E_0\sqrt{\pi}}{2\omega L}\, e^{-Rt/2L} \int_0^t x^{1-2\rho}\, e^{Rx/2L} \sin\{\omega(t-x)\}\, H[zx^{-2h}]\, dx. \qquad \ldots(2.10)$$

Now appealing to a property of the H-function [Mathai and Saxena 1978, p. 4, eqn. (1.2.2)] and evaluating the integral with the help of (1.1) [or (1.3)] with $c = 1$, b, $\epsilon \to 0$, we get the following value of $q(t)$:

$$q(t) = J(t) + \frac{E_0(t)^{3-2\rho}\sqrt{\pi}}{2L}\, e^{-Rt/2L} \sum_{r,k=0}^{\infty} \frac{(-\omega^2t^2)^k (Rt/2L)^r}{r!}$$

$$\times\ H^{M+1,N}_{P+1,Q+1}\left[zt^{-2h} \,\middle|\, \begin{matrix}(a_j, A_j)_{1,P},\ (4-2\rho+r+2k, 2h)\\ (2-2\rho+r,\ 2h),\ (b_j, B_j)_{1,Q}\end{matrix}\right] \qquad \ldots(2.11)$$

$$= J(t) + \frac{E_0(t)^{3-2\rho}\sqrt{\pi}}{2L} \sum_{r=0}^{\infty} (r+1)\, B_r(-Rt/2L)^r$$

$$\times\ H^{M+1,N}_{P+1,Q+1}\left[zt^{-2h} \,\middle|\, \begin{matrix}(a_j,A_j)_{1,P}, (4-2\rho+r,\ 2h)\\ (2-2\rho,\ 2h),\ (b_j,\ B_j)_{1,Q}\end{matrix}\right] \qquad \ldots(2.12)$$

where $J(t)$ is given by (2.3) and

$$B_r = \sum_{k=0}^{[r/2]} \frac{(-1)^k (-r)_{2k} (2L\omega/R)^{2k}}{(2)_{2k}}. \qquad \ldots(2.13)$$

The conditions of validity of (2.11) and (2.12) are

(i) $h > 0$, $t > 0$, $A > 0$, $|\arg z| < (\tfrac{1}{2}) A\pi$ {A is given by (1.4)}.

(ii) $\mathrm{Re}(1-\rho) + h \min\limits_{1\le j\le N} \{\mathrm{Re}(1-a_j)/A_j\} > 0$.

Thus, when the electromotive force $E_0P(t)$ is in the form of series of sines:

$$E_0 \sum_{r=0}^{\infty} H^{M+1,N+1}_{P+2,Q+2}\left[z \,\middle|\, \begin{matrix}(1-\rho-r,\ h),\ (a_j,A_j)_{1,P},\ (2-\rho+r,\ h)\\ (\tfrac{3}{2}-\rho,\ h),\ (b_j,B_j)_{1,Q}, (1-\rho,\ h)\end{matrix}\right]$$

$$\times \sin(2r+1)\, t \qquad \ldots(2.14)$$

the value of $q(t)$ is given by (2.11) {or (2.12)}.

The solution of (2.1) given by Gupta and Goyal [(1973), p. 738, eqn. (3.4)] is contained in our solution (2.11). This can be verified easily by putting $R = 0$, $\rho = 0$, $h = 1$, $A_i = B_j = 1 (i = 1,\ldots, P\,;\; j = 1,\ldots,Q)$ in (2.11) and appealing to the Gamma multiplication formula therein.

3. Some Interesting Special Cases

The solutions of (2.1) are quite general in character as these possess twofold generality. The first one is the general nature of the H-function and second is exhibited by the presence of the general arguments in this function. By making a free use of results (Mathai and Saxena 1978, pp. 145 to 151), our solutions can be suitably applied to a remarkably wide variety of useful functions (or product of such functions) that occur frequently in the problems of mathematical physics and engineering. Here we mention only some interesting special cases of the solution of (2.1) given by (2.6).

On taking $c = 1$ and $b \to 0$ in (2.5) and (2.6), we find that

$$P(t) = t^{\rho-1}\, H^{M,N}_{P,Q}\left[zt^{u} \,\middle|\, \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix} \right] \qquad \ldots(3.1)$$

then

$$q(t) = J(t) + \frac{E_0}{L}\,(t)^{\rho+1}\, e^{-Rt/2L} \sum_{r,k=0}^{\infty} \frac{(-\omega^2 t^2)^k\,(Rt/2L)^r}{r!}\,.$$

$$\times\, H^{M,N+1}_{P+1,Q+1}\left[zt^{u} \,\middle|\, \begin{matrix} (1-\rho-r,\, u),\, (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q},\, (-1-2k-\rho-r,\, u) \end{matrix} \right] \qquad \ldots(3.2)$$

or, equivalently,

$$q(t) = J(t) + \frac{E_0}{L}\,(t)^{\rho+1} \sum_{r=0}^{\infty} (r+1)\; B_r(-Rt/2L)^r$$

$$\times\, H^{M,N+1}_{P+1,Q+1}\left[zt^{u} \,\middle|\, \begin{matrix} (1-\rho,\, u),\, (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q},\, (-1-\rho-r, u) \end{matrix} \right] \qquad \ldots(3.3)$$

where $J(t)$ and B_r are given by (2.3) and (2.13) respectively. The conditions of validity for the solution can be easily obtained from those given after (1.3).

Further specializing the parameters of the H-functions involved in (3.1) and (3.3) on the lines indicated by Mathai and Saxena (1978, pp. 151, 152 and 154), we get the following useful and interesting results after a little simplification:

Equation	$E_0P(t)$	$q(t)$
(3.4)	$E_0t^{\rho-1}e^{-zt}$	$J(t) + \dfrac{E_0t^{\rho+1}}{L}\sum_{r=0}^{\infty}\dfrac{(r+1)B_r\Gamma(\rho)}{\Gamma(\rho+r+2)} \times {}_1F_1[\rho;\ \rho+r+2;\ -zt]\ (-Rt/2L)^r$
(3.5)	$E_0t^{\rho-1}(z+t)^{-a}$	$J(t) + \dfrac{E_0t^{\rho+1}\Gamma(\rho)}{Lz^a}\sum_{r=0}^{\infty}\dfrac{(r+1)B_r}{\Gamma(\rho+r+2)} \times {}_2F_1[\rho, a;\ \rho+r+2;\ -t/z]\ (-Rt/2L)^r$
(3.6)	$E_0t^{\rho-1}\sin(zt)$	$J(t) + \dfrac{E_0t^{\rho+2}}{L} \times z\sum_{r=0}^{\infty}\dfrac{\Gamma\left(\frac{\rho+1}{2}\right)\Gamma\left(\frac{\rho+2}{2}\right)(r+1)B_r}{2^{r+2}\,\Gamma\left(\frac{\rho+r+3}{2}\right)\Gamma\left(\frac{\rho+r+4}{2}\right)} \times {}_2F_3\left[\frac{\rho+1}{2}, \frac{\rho+2}{2};\ \frac{3}{2}, \frac{\rho+r+3}{2}, \frac{\rho+r+4}{2};\ \frac{-z^2t^2}{4}\right] \times (-Rt/2L)^r$
(3.7)	$E_0t^{\rho-1}e^{-zt/2}W_{\lambda,m}(zt)$	$J(t) + \dfrac{E_0\,t^{\rho+3/2}}{L}\sqrt{z}\sum_{r=0}^{\infty}(r+1)Br \times G_{2,3}^{2,1}\left[zt\,\middle\vert\,\begin{matrix}-\rho+\frac{1}{2}, -\lambda+\frac{1}{2}\\ \pm m, -\rho-r-3/2\end{matrix}\right](-Rt/2L)^r$

where $J(t)$ and B_r are defined by the eqns. (2.3) and (2.13), respectively.

Finally, we remark that the solution of (1.1) for the values of $P(t)$ given in the above table with $R = 0$ do not appear to follow from it, due to the presence of R in the denominator of B_r. In this case, we should take eqn. (3.2) as the solution of (2.1) and refer to Gupta and Goyal (1973).

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