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REAL ROOTS OF RANDOM HARMONIC EQUATIONS

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Let $c_0, c_1, c_2, \ldots, \ldots, c_n$; be a sequence of mutually independent normally distributed random variables with mathematical expectation zero and variance unity. Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots, \ldots, \phi_n(x)$, be a sequence of classical orthonormal polynomials and $a_0, a_1, a_2, \cdots, \cdots, a_n$; a sequence of constants which when multiplied by $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$, in order, normalise them over the fundamental interval of the harmonic functions $\phi_k(x)$. It is proved that the mathematical expectation of the real zeros of $\sum_{k=0}^{n} c_k a_k \phi_k(x)$, which lie in the fundamental interval is equal with $n/\sqrt{3}$, asymptotically as $n \to \infty$.

Let $c_0, c_1, c_2, \ldots, \ldots$ be a sequence of mutually independent, normally distributed random variables with mathematical expectation zero and variance unity. Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ be a sequence of real valued polynomials (functions) and $a_0, a_1, a_2, \ldots$, a sequence of real constants. The real roots of the equation

$$c_0 a_0 \phi_0(x) + c_1 a_1 \phi_1(x) + c_2 a_2 \phi_2(x) + \ldots + c_n a_n \phi_n(x) = 0,$$

have been considered by various authors. J. E. Littlewood and A. C. Offord had shown, in the year 1939, that when $a_k = 1$, $\phi_k(x) = x^k$ and n is large most of the equations have atmost 25 $(\log n)^2$ real roots. When $a_0 = 0$, $a_k = 1$ $(k \neq 0)$, $\phi_k(x) = \cos k (\cos^{-1} x)$, J.E.A. Dunnage has shown that most of the equations posses nearly $2n/\sqrt{3}$ roots in the interval $(-1, 1)$, when n is large.

Now the x^k's are a set of functions monotonic in $(-\infty, 0]$ and $[0, \infty)$, whereas $\cos k (\cos^{-1})$ oscillate k-times between -1 and $+1$. This raises the question as to how far the oscillatory nature of $\phi_k(x)$ is transferred into the sum $\Sigma\, c_k a_k \phi_k(x)$, or what is the same thing how many times the last sum passes through zero, the mean value of $\Sigma\, c_k a_k \phi_k$ as x varies over a prescribed interval. In this work we consider the cases when the ϕ_k's are the classical orthogonal polynomials over some interval (bounded or not), since the oscillatory natures are very accurately known. We hope that our results here will substantially remain the same when the coefficients belong to domain of attraction of stable law.

When the interval of definition is changed into one of the three standard intervals $[-1, 1]$, $[0, \infty)$, and $(-\infty, \infty)$, the classical polynomials bear the names Jacobi, Laguerre and Hermite. There exists weight function $\omega(x)$, corresponding to the interval concerned, over which the integral $\omega(x)\, \phi_k^2(x)$ is positive number g_k, and we take $a_k = g_k^{-1/2}$. The integral of $\psi_k^2(x) = a_k^2 \phi_k^2(x)$, over the corresponding interval, is unity so that each term of $\Sigma\, c_k \psi_k(x) = c_k a_k \phi_k(x)$, has the same weightage in the same sense. We shall show that in each case, when n is large, the set of equations may be expected to have at least $c.n$ real roots ($c>0$), i.e. the oscillatory property of $\psi_k(x)$ is also shared by the sum $\Sigma\, c_k\, \psi_k(x)$. Indeed, most of the sums of the last type also perform a non-zero portion of the maximum number of oscillations possible for $\psi(x)$, as x varies over the fundamental interval once only.

We shall denote the number of real roots of $\sum_{k=0}^{n} c_k\, \psi_k\, (x) = 0$ in the interval (a, b) as $N_n\, (.;a,b)$ Das (1971) has considered the case when $\psi_k\,(x) = a_k P_k(x) = P_k{}^*(x)$, the normalised Legendre polynomials. He has derived the formula for the mathematical expectation of $N_n\, (.\,;\, a,\, b)$ in the form

$$E\, N_n\,(\cdot\,;\, a,\, b) = \frac{1}{\pi}\int_a^b f_n\,(x)\, dx,$$

$$\text{where } f_n^2\,(x) = \frac{S_n\,(x) + R_n\,(x)}{D_n\,(x)} - \frac{1}{4}\,\frac{Q_n^2\,(x)}{D_n^2(x)} \qquad \ldots (1)$$

with $D_n\,(x) = \phi'_{n+1}(x)\, \phi_n(x) - \phi_{n+1}(x)\phi'_n(x) = \Sigma\, \phi_k^2\,(x) > 0$

$$Q_n\,(x) = \phi''_{n+1}\,(x)\phi_n\,(x) - \phi_{n+1}(x)\, \phi'_n(x)$$

$$R_n\,(x) = \tfrac{1}{2}\left[\, \phi''_{n+1}\,(x)\, \phi'_n\,\,(x) - \phi'_{n+1}\,(x)\, \phi''_n\,(x)\right]$$

$$\text{and } S_n\,(x) = \frac{1}{6}\left[\, \phi'''_{n+1}(x)\, \phi_n(x) - \phi_{n+1}(x)\, \phi'''_n(x)\,\right].$$

The cases of Jacobi, Hermite and Laguerre polynomials will be considered in the sections 2, 3 and 4 respectively.

§2. For $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ form an orthogonal set over $(-1, 1)$ with weight function $(1-x)^\alpha\, (1+x)^\beta$ the g_k being determined by (Rainville 1960, p. 260).

To compute the $D_n\,(x)$, etc. we shall use the relations (writing $y_n = y_n\,(x)$) for $P_n{}^{(\alpha,\beta)}\,(x)$, (Erdelyi *et al.* 1953, p. 169).

$$(1-x^2)\, y''_{n+1} = \left[\alpha - \beta + (\alpha+\beta+2)\, x\right]\, y'_{n+1} - (n+1)\,(n+2+\alpha+\beta)\, y_{n+1} \qquad \ldots(2)$$

$$(1-x^2)\, y''_n = \left[\alpha-\beta+(\alpha+\beta+2)\, x\right] y'_n - n\,(n+1+\alpha+\beta)_n\, y. \qquad \ldots(3)$$

Multiplying (2) by $y_n^{(k)}$ and (3) by $y_{n+1}^{(k)}$ and subtracting, we obtain for $k = 1,\ 0,$ respectively the relations

$$(1-x^2)\left(\, y''_{n+1}\, y'_n - y'_{n+1}\, y'_n\right) = n\,(n+1+\alpha+\beta)\left(y'_{n+1}\, y_n - y_{n+1}\, y''_n\right)$$

$$- (2n+2+\alpha+\beta)\, y_{n+1}\, y'_n \qquad \ldots(4)$$

$$(1-x^2)\left(y''_{n+1}\,y_n-y''_n\,y_{n+1}\right)=\left[\alpha-\beta+(\alpha+\beta+2)x\right]\left(y'_{n+1}\,y_n-y'_n\,y_{n+1}\right)$$
$$-(2n+2+\alpha+\beta)\,y_n\,y_{n+1}. \qquad ...(5)$$

If we differentiate (2) and (3) we get y'''_{n+1} and y'''_n. Multiplication by y_n and y_{n+1} and subsequent subtraction yields, after some simplification the relation

$$(1-x^2)\left(y'''_{n+1}\,y_n-y_{n+1}\,y''_n\right)=A_n(x)\left(y'_{n+1}\,y_n-y_{n+1}\,y'_n\right)$$
$$-B_n(x)\,y_{n+1}\,y_n-c_n(x)\,y_n^2 \qquad ...(6)$$

where
$$(1-x^2)\,A_n(x)=\left[\alpha-\beta+(\alpha+\beta+4)\,x\right]\left[\alpha-\beta+(\alpha+\beta+2)x\right]$$
$$-n\,(n+1+\alpha+\beta)+\alpha+\beta+2$$
$$(1-x^2)\,B_n(x)=(2n+2+\alpha+\beta)\left[\alpha-\beta+(\alpha+\beta+4)x\right]+(n+1)$$
$$\left[\alpha-\beta-(2n+2+\alpha+\beta)x\right](1-x^2)\,c_n(x)=2\,(n+1+\alpha)\,(n+1+\beta).$$

For large n, we shall use asymptotic formula for y_n as given in (Askey and Wainger 1968, p. 33) namely

$$\left\{\Gamma(\tfrac{1}{2})\Gamma(n+1)\Big/\Gamma(n+\tfrac{1}{2})\right\}(\sin\theta/_2{}^{\alpha+1/2}\,(\cos\theta/_2)^{\beta+1/2}\,P_n^{(\alpha,\beta)}\,(\cos\theta)$$
$$=\left[1-\frac{(\alpha+\frac{1}{2})\,(\beta+\frac{1}{2})}{2n-1}\right]\cos\left[(n+\frac{\alpha+\beta+1}{2})\,\theta-\frac{\pi}{2}\,(\alpha+\tfrac{1}{2})\right]$$
$$-\left[\frac{\alpha^2-\frac{1}{4}}{2(2n-1)}\cot\theta/2-\frac{\beta^2-\frac{1}{4}}{2(2n-1)}\tan\theta/2\right]\times\sin\left[(n+\frac{\alpha+\beta+1}{2})\,\theta\right.$$
$$\left.-\frac{\pi}{2}\,(\alpha+\tfrac{1}{2})\right]+O\left((n\sin\theta)^{-2}\right).$$

On taking $x=\cos\theta$, $\alpha+\beta+1=2B$, $n\,(2c+1)=-4c$ we have for large in

$$y_n(x)=\left(\frac{2^{\alpha+\beta+1}}{\pi n}\right)^{\frac{1}{2}}\left(1-x\right)^{-\frac{\alpha}{2}-\frac{1}{4}}\left(1+x\right)^{-\beta/2-\frac{1}{4}}$$
$$\left[\cos(n\theta+B\theta+c)+O(1/_n\sin\theta)\right]. \qquad ...(7)$$

Let $\gamma=\max(0,\alpha,\beta)\geqslant 0$, $0<\epsilon'<2/(3+2\gamma)$.

The differential recurrence relations for the Jacobi set may be written in the form

$$(1-x^2)y'_n=n\left[(\alpha-\beta)\,(2n+\alpha+\beta)^{-1}-x\right]y_n+2\,(n+\alpha)(n+\beta)(2n+\alpha+\beta)^{-1}y_{n-1}$$

$$(1-x^2)\ y'_{n+1} = (n+1)\left[(\alpha-\beta)(2n+2+\alpha+\beta)^{-1} - x\right] y_{n+1} + 2\,(n+1+\alpha)$$
$$(n+1+\beta)(2n+2+\alpha+\beta)^{-1}\ y_n$$

This makes D_n expressed by the relation

$$(1-x^2)\ D_n = (1-x^2)\ (y'_{n+1}y_n - y'_n\ y_{n+1}) = [-x+(\alpha^2-\beta^2)(2n+\alpha+\beta)^{-1}(2n+\alpha+\beta+2)^{-1}]$$
$$y_n y_{n+1} + 2\,(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)^{-1}\ (y_n^2 - y_{n+1}y_{n-1})$$
$$+2\,(2n+\alpha+\beta+1)(2n+\alpha+\beta)^{-1}\ (2n+\alpha+\beta+2)^{-1}\ y^2{}_n - 4\,(n+\alpha)(n+\beta)$$
$$(2n+\alpha+\beta)^{-1}\ (2n+\alpha+\beta+2)^{-1}\ y_n\ y_{n+1}$$

and using the asymptotic relation (7) for large n, we obtain estimate $D_n(x) \sim 2^{\alpha+\beta+1}\ \pi^{-1}\ (1-x)^{-\alpha-1/2}\ (1+x)^{-\beta-\frac{1}{2}}$

$$\{1-x^2+O(1/n\ \sin\theta)\}/(1-x^2) \qquad ..\ (8)$$

valid in the range $-1+n^{-\epsilon} \leqslant x \leqslant 1-n^{-\epsilon'}$. Also in this range

$$|y_{n+1}\ y_n| = 2^{\alpha+\beta+1}(\pi n)^{-1}\ (1-x)^{-\alpha-1/2}\ (1+x)^{-\beta-1/2}\ |\cos(x+\theta)\cos x + O(1/n\ \sin\theta|$$
$$\leqslant (A/n)\ (1-x)^{-\alpha-1/2}\ (1+x)^{-\beta-1/2} \qquad ..\ (9)$$

for an absolute constant A, where we have put x for $n\theta+B\theta+c$ for convenience. Similarly

$$y_n^2 \leqslant (A/n)\ (1-x)^{-\alpha-1/2}\ \ (1+x)^{-\beta-1/2}. \qquad ...(10)$$

Writing $\rho = \left(2^{\alpha+\beta+1}/\pi n\right)^{\frac{1}{2}}\left(1-x\right)^{-\alpha/2-1/4}\left(1+x\right)^{-\beta/2-1/4}$

we have from the asymptotic relation (7) with $x = n\theta+B\theta+c$ the estimate

$$(1-x^2)\ y'_n\ y_{n+1} = n\,[(\alpha-\beta)(2n+\alpha+\beta)^{-1} - x]\ y_n\ y_{n+1}$$
$$+2\,(n+\alpha)(n+\beta)(2n+\alpha+\beta)^{-1}\ y^2{}_{n+1}$$
$$\sim \left(\frac{\alpha-\beta}{2} - nx\right)\rho^2\ \left\{\cos x + O\left(\frac{1}{n\ \sin\theta}\right)\right\}\left\{\cos(x+\theta) + O\left(\frac{1}{n\ \sin\theta}\right)\right\}$$
$$+\ np^2\left\{\cos(x+\theta) + O\left(\frac{1}{n\sin\theta}\right)\right\}\left\{\cos(x-\theta) + O\left(\frac{1}{n\ \sin\theta}\right)\right\}$$
$$\sim \rho^2 n\left\{\cos(x+\theta) + O\left(\frac{1}{n\sin\theta}\right)\right\}\left\{\cos(x-\theta) - \cos\theta\ \cos x + O\left(\frac{1}{n\ \sin\theta}\right)\right\}$$
$$\sim \rho^2 n\left\{\cos(x+\theta) + O(1/\sin\theta)\right\}\left\{\sin x\ \sin\theta + O\,(1/n\ \sin\theta)\right\}.$$

Since $\sin x = (1-x^2)^{1/2}$, this makes

$$|y'_n y_{n+1}| < A\,(1-x)^{-\alpha}\ (1+x)^{-\beta}\ (1-x^2)^{-1}. \qquad ...(11)$$

Now we shall estimate the expression (1) for $|x \pm 1| > n^{-\epsilon'}$. We find from (4) (8) and (11) that

$$\{D_n(x)\}^{-1}\ R_n(x) = \frac{n(n+1+\alpha+\beta) - (2n+2+\alpha+\beta)\ y_{n+1}\ y'_n\ D_n^{-1}(x)}{2(1-x^2)}$$
$$= \frac{n^2}{2\,(1-x^2)} + O(n/1-x^2) + O\left(n/(1-x^2)^{3/2}\right) + O\left(1/(1-x^2)^3\right). \qquad ...(12)$$

Similarly from (6), (9) and (10)

$$\left\{ D_n(x) \right\}^{-1} S_n(x) = \frac{[\alpha-\beta+(\alpha+\beta+4)x]\ [\alpha-\beta+(\alpha+\beta+2)x]}{6(1-x^2)}$$

$$+ \frac{\alpha+\beta+2-n(n+1+\alpha+\beta)}{6(1-x^2)} - \frac{1}{6(1-x^2)} \frac{y_n^2}{D_n(x)} 2(n+1+\alpha)\ (n+1+\beta)$$

$$- \frac{1}{6\,(1-x^2)} \frac{y_n\, y_{n+1}}{D_n\,(x)} \left[\left\{(2n+2+\alpha+\beta)\ (\alpha-\beta+(\alpha+\beta+4)\,x\right\}\right] + (n+1)$$

$$\left[\alpha-\beta+(2n+2+\alpha+\beta)\,x\right]$$

$$= -\frac{n^2}{-6(1-x^2)}\left\{1+O\left(\frac{1}{n}\right)\right\} + O\left(\frac{n}{1-x^2}\right) + O\left(\frac{1}{(1-x^2)^{5/2}}\right)\Bigg\}.$$

Hence $D_n^{-1}(x)\left\{R_n(x)+S_n(x)\right\} = \dfrac{n^2}{3(1-x^2)} + O\left(\dfrac{n}{(1-x^2)^{3/2}}\right) + O\left(\dfrac{1}{(1-x^2)^3}\right)$. ...(13)

Further, in the above range for x, we have from (8) and (5)

$$\frac{1}{4} D_n^{-1}(x)\, Q_n^2(x) = \left[\frac{\alpha-\beta+(\alpha-\beta+2)\,x}{2\,(1-x^2)} - \frac{n y_n y_{n+1}\left\{1+O\left(\frac{1}{n}\right)\right\}}{2\,D_n\,(x)}\right]^2$$

$$= \left[O\left(\frac{1}{1-x^2}\right) + O(1) + O\left\{\frac{1}{n}\,\frac{1}{(1-x^2)^{3/2}}\right\}\right]^2$$

$$= O\,(1/1-x^2) + O\,(1/n^2\,(1-x^2)^3) + O\,(1/_n\,(1-x^2)^{5/2}). \qquad ...(14)$$

From (13) and (14) we obtain for x in the above range,

$$D_n^{-1}(x)\ \{R_n(x)+S_n(x)\} - \tfrac{1}{4}\ D_n^{-2}(x)\ \ Q_n^2(x)$$

$$= \frac{1}{3}\ \frac{n^2}{1-x^2}\left[1+O\left(\frac{1}{n(1-x^2)^{1/2}}\right) + O\left(\frac{1}{n^2(1-x^2)^2}\right) + O\left(\frac{1}{n^3(1-x^2)^{3/2}}\right)\right].$$

Let $\eta = n^{-1/2+\gamma}$. Then for $-1+\eta \leqslant x \leqslant 1-\eta$, we find

$$f_n(x) = \frac{1}{\pi}\left[D_n^{-1}(x)\,\{R_n(x)+S_n(x)\} - \frac{1}{4}\ D_n^{-2}(x)\ Q_n^2(x)\right]^{1/2}$$

$$= \frac{1}{\pi}\ \frac{1}{\sqrt{3}}\ \frac{n}{(1-x^2)^{1/2}}\left[1+O\,(n^{-(3+2\gamma)/(4+2\gamma)}\right].$$

$$\text{Thus } EN_n\,(-1+\eta,\ 1-\eta) = \int_{-1+\eta}^{1-\eta} \frac{1}{\pi}\ \frac{1}{\sqrt{3}}\,\frac{n}{(1-x^2)^{1/2}}\ [1+O\,(n^{-(3+3\gamma)/(4+2\gamma)}]\,dx$$

$$= \frac{1}{\sqrt{3}}\,n + O\,(n^{1/(4+2\gamma)})$$

where $\gamma = \max\ (0, \alpha, \beta) \geqslant 0$.

We shall show that in the ranges $1-\eta \leqslant x \leqslant 1$ and $-1 \leqslant x \leqslant -1+\eta$ the number of zeros of $\Sigma\ c_k a_k \phi_k(x)$ is, with probability at least $1-2/n$, equal with $An^{1/3}$.

To this end we put $f(x)=f(\bar{c}, x)=\Sigma\, c_k a_k \phi_k(x)$,
where $\bar{c}$ denotes the random vector $(c_0, c_1, \ldots, c_n)$. Now $f(\bar{c}, 1)=\Sigma c_k a_k \phi_k(1)$ is a random variable with distribution function

$$\frac{1}{(2\pi\Lambda^2)^{1/2}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\frac{n^2}{\Lambda^2}\right) du$$

where $\Lambda^2 = c_0^2\, a_0^2\, \phi_0^2(1) + \ldots + a_n^2\, \phi_n^2(1) \geqslant \text{Min}\ \{a_0^2\, \phi_0^2(1), \ldots, a_n^2\, \phi_n^2(1)\}$

$$\text{so that } P_r\left(\,|f(1)| < \exp(-2n\epsilon)\right) = \left(\frac{2}{\pi\Lambda^2}\right)^{\frac{1}{2}} \int_0^{e^{-n\epsilon^2}} \exp\left(-\frac{u^2}{2\Lambda^2}\right) du$$

$$\leqslant \left(\frac{2}{\pi\Lambda^2}\right)^{1/2} e^{-2n\epsilon} < e^{-n\epsilon}$$

where $\epsilon^2=\eta$. If $h_n=\max_{0\leqslant k\leqslant n} |c_k|$, then $|f(1+2\epsilon e^{i\theta})| \leqslant h_n \{\Sigma a_n\, |\phi_k(1+2\epsilon e^{i\theta})|\}$.

Now we can show as in [Erdelyi *et al.* 1953, p. 722] that $P_\gamma(h_n \leqslant n) \geqslant 1-3e^{-n^2/2}$

Let $M_n = \underset{0\leqslant k\leqslant n}{\text{Max}} \left|a_k\phi_k(1+2\epsilon e^{i\theta})\right|$ so that

$P_\gamma\{\,|f(1+2\epsilon e^{i\theta})| \leqslant n^2 M_n\} \geqslant 1-3e^{-n^2/2}$.

By the Schafli representation of Jacobi polynomials (Erdelyi *et al.* 1953, p. 172)

$$P_k^{(\alpha,\beta)}(\zeta) = \frac{1}{2\pi i} \oint(\xi+) \left(\frac{t^2-1}{2}\right)^k \left(\frac{1-t}{1-\zeta}\right)^\alpha \left(\frac{1+t}{1-\zeta}\right)^\beta dt/(t-\xi)^{k+1}$$

for a simple closed contour (ζ^+) around ζ in the positive sense, the contour being so taken that it has the points -1 or $+1$ neither in the interior nor on the boundary. After some simplification we find

$$P_k^{(\alpha,\beta)}(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} (\zeta+i\sqrt{1-\zeta^2}\cos\phi)^k \left(1-i\sqrt{\frac{1+\zeta}{1-\zeta}}\, e^{i\phi}\right)^\alpha$$

$$\left(1+i\sqrt{\frac{1-\zeta}{1+\zeta}}\, e^{i\phi}\right)^\beta d\phi$$

so that, remembering that

$n^{-1/2+\gamma} = \eta = \epsilon^2$ we have

$$\left|P_k^{(\alpha,\beta)}(1+2\epsilon\, e^{i\theta})\right| \leqslant (1+2\epsilon)^k \left(1+\frac{2}{\epsilon}\right)^\alpha (1+2\epsilon)^\beta$$

and thus

$$M_n \leqslant (1+2\epsilon)^n \left(1+\frac{2}{\epsilon}\right)^\alpha \left(1+2\epsilon\right)^\beta \leqslant \exp\left[2n^{(3+2\gamma)/(4+2\gamma)}\right].\, n^A$$

where A is a constant depending on α alone. Thus with probability at least

$$1-3\exp(-n^2/2) - \exp(-n\epsilon) > 1 - \frac{1}{n}$$

$$\left|\frac{f(1+2\eta e^{i\theta})}{f(1)}\right| \leqslant n^{2+A} \exp\left(n^{(3+2\gamma)/(4+2\gamma)} + n\eta\right).$$

By an application of Jensen's theorem, we find that (cf. Titchmarsh 1939).

$$\nu\left(\bar{c},2\eta\right)\leqslant\left(2\pi\log 2\right)^{-1}\int_{0}^{2\pi}\log\left|\frac{f(\bar{c},\,1+2\eta e^{i\theta})}{f(\bar{c}\,,\,1)}\right|d\theta=O\left(n^{3+2\gamma/(4+2\gamma)}\log n\right)$$

with probability at least $1-1/n$. Combining with the results in §2 we find that

$$EN_n=\frac{n}{\sqrt{3}}+O\left(n^{(3+2\gamma)/(4+2\gamma)}\log n\right)$$

as $n\rightarrow\infty$. We state the above result in our theorem:

Theorem — If $\psi_k(x)$ denote the normalised Jacobi polynomials over the interval $(-1, 1)$ of degree k in x, then the number of real zeros of $\sum_{k=0}^{n} c_k\,\psi_k(x)$ has the expected density $f_n(x)$ at x where $f_n(x)=\frac{n}{\pi\sqrt{3}}\;\frac{1}{(1-x^2)^{1/2}}\left(1+O(n^{-1/(4+2\gamma)})\right)$ for any x in the range $-1+n^{-1/2+\gamma}\leqslant x\leqslant 1-n^{-1/2+\gamma}$, $(n\rightarrow\infty)$.

The average number of real roots of $\sum_{k=0}^{n} c_k\,\psi_k(x)=0$ which lie in $-1\leqslant x\leqslant 1$ is $\frac{n}{\sqrt{3}}+0\left(n^{3+2\gamma/4+2\gamma}\log n\right)$ when $n\rightarrow\infty$. Here $\gamma=\max(0,\alpha,\beta),\geqslant 0$.

§3. The Hermite polynomials $H_k(x)$ form an orthogonal set over the entire x-axis $(-\infty<x<\infty)$ with respect the weight function $\exp(-x^2)$. The $H_k(x)$ are expressible as $H_k(x)=\exp(x^2)\,D^k\exp(-x^2)$ (cf. Erdelyi *et al.* 1955, p. 193). Taking $a^2{}_n=(\pi^{1/2}\,2^n\,n!)^{-1}$, we get the normalised Hermite polynomials $\psi_k(x)=a_kH_kx=a_k\phi_k(x)$ in the notations of the section 1. To estimate the various expressions $D_n(x)$ etc. we shall use the differential-recurrence relations for the set, namely

$$H_{n+1}(x)-2x\,H_n(x)+2n\,H_{n-1}(x)=0 \qquad \ldots(15)$$

$$H''_{n+1}=2x\,H'_{n+1}-2(n+1)\,H_{n+1} \qquad \ldots(16)$$

$$H''_n=2x\,H'_n-2n\,H_n \qquad \ldots(16\text{A})$$

and on differentiation we further have

$$H'''_{n+1}=2x\,H''_{n+1}-2n\,H'_{n+1} \qquad \ldots(17)$$

$$H'''_n=2x\,H''_n-2(n-1)\,H'_n \qquad \ldots(17\text{a})$$

$$\text{and } D_n=H'_{n+1}\,H_n-H'_n\,H_{n+1}\,. \qquad \ldots(18)$$

From these we easily obtain

$$H''_{n+1}\,H'_n-H''_n\,H'_{n+1}=2nD_n\,2H'_nH_{n+1} \qquad \ldots(19)$$

$$H''_{n+1}\,H_n-H''_n\,H_{n+1}=2xD_n-2H_n\,H_{n+1} \qquad \ldots(20)$$

$$H'''_{n+1} H_n - H'''_n H_{n+1} = 2x(H''_{n+1} H_n - H''_n H_{n+1}) - 2n(H'_{n+1} H_n - H'_n H_{n+1})$$
$$- 2H'_n H_{n+1} = (4x^2 - 2n) D_n - 4x H_n H_{n+1} - 2H'_n H_{n+1} \,. \qquad \ldots(21)$$

To obtain estimates for $f_n(x)$, the expected density for the zeros $\sum_{k=0}^{n} c_k \psi_k(x)$ at x when n is large, we divide the x-axis into a number of parts. This is because there is no single asymptotic estimate of $H_n(x)$ etc. for all x and complexity arises for values of x comparable with a power of n.

(i) "Near the origin" for $x = O(n^{1/6-\epsilon})$, we have [Sansona 1959, p. 327]

$$l_n \exp(-x^2/2)\, H_n(x) = \cos(N^{1/2}x - n\pi/2) + O(x^3 N^{-1/2})$$

with $N = 2n+1$ and $l_n \sim \Gamma(-\frac{n}{2}+1) \big/ \Gamma(+n1)$. Thus H_n and H_{n+1} cannot vanish simultaneously. Now $-2 H'_n H_{n+1}/D_n(x) = 2$, for such x for which $H_n(x)$ is zero. If $H_n x \neq 0$, then the ratio equals A where $|A| \leqslant 2$, by the above asymptotic relations. Further the same relations ensure $(-2H_n H_{n+1}/D_n) \sim \frac{1}{n}$. Thus, we have from (19), (20) and (21) (retaining the notations Q_n, R_n and S_n when ϕ_k is replaced by H_k in (1)), the estimates $R_n \sim nD_n$, $S_n \sim \frac{1}{6}(4x^2 - 2n)D_n$ $Q_n \sim 2x\, D_n$ so that

$$f_n(x) = \frac{1}{\pi}\sqrt{\frac{R_n + S_n}{D_n} - \frac{Q_n}{4D_n^2}} \sim \sqrt{\frac{2n - x^2}{\sqrt{3}\,\pi}} \qquad \ldots(22a)$$

(ii) "In the oscillatory region distant from zero" for $n^{1/6+\epsilon} < |x| < \mu(2n)^{1/2}$ with $0 < \mu < 1$, the relations [Erdélyi *et al.* 1953, p. 193&200]

$$H_{2m}(x) = (-1)^m 2^{2m} m!\, L_m^{-1/2}(x^2) \quad \text{if} \quad n = 2m$$
$$H_{2m+1}(x) = (-1)^m 2^{2m+1} m!\, L_m^{\frac{1}{2}}(x^2) \quad \text{if} \quad n = 2m+1$$
$$\exp(-x^2/2)\; L_m^{\alpha}(x^2) = (-2)^m (2\cos\theta)^{-\alpha} (\pi\nu\sin 2\theta)^{1/2} \left(1 + O(\frac{1}{m})\right)$$

where $\nu = \nu(\alpha) = 4m + 2\alpha + 2$ can be used. However in this range of length $O(n^{1/2})$ most of the n zeros of H_n and H_{n+1} are concentrated [Sansona 1959, p. 313-15]. Indeed the $[n/2]$ zeros of $H_n(x)$ can be arranged as $0 < x_{1,n} < x_{2n} < \ldots < x\left[\frac{n}{2}\right], n < \sqrt{2n+1}$,

where $\nu = 0,1,2, \ldots \left[\frac{n}{2}\right] \nu\pi/\sqrt{2n+1} < x_{n\nu} < (4\nu+3)/\sqrt{2n+1}$ if n is odd,

$(\nu - \frac{1}{2})\,\pi/\sqrt{2n+1} < x_{\nu,n} < (4\nu+1)/\sqrt{2n+1}$ if n is even, so that 3ν zeros lie in the interval of length $15\nu/\sqrt{2n+1}$, giving a density $O(\sqrt{2n}/5)$ (per unit length) in this region. If $H_n(x) = 0$, then the three term recurrence relation (15) ensures $H_{n+1} \neq 0 \neq -2n\, H_{n-1}$ so that $D_n = (2nH_{n-1})^2 = H^2_{n+1} = -H'_n H_{n+1}$, $Q_n = 2xD_n$,

$$R_n = (n+1)\, D_n, \quad S_n = \tfrac{1}{3}(2x^2 - n + 1)\, D_n.$$

If $H_{n+1}(x) = 0$, $H_n(x) = (n/x\, H_{n-1}) \neq 0$, $D_n = 2(n+1) H^2_n$ and the remaining equalities hold good. If $H_{n-1}(x) = 0 = H'_n(x)$

$$D_n = 2\,(n+1)\; H^2{}_n, \quad H_{n+1} = 2xH_n$$

and the remaing equalities hold good atleast asymptotically.

Thus if $|x| \leqslant \sqrt{2n+1} - 4$ and one of H_{n-1}, H_n and H_{n+1} vanishes, then

$$f_n^2(x) \;\geqslant\; \frac{1}{\pi^2}\;\frac{2n-x^2}{3\theta}\;, \qquad (\theta \geqslant 1). \qquad \text{...(22a)}$$

Now $$f_n^2(x) = \frac{1}{\pi^2}\sum_{r<s}\left(\psi_r\,\psi_s' - \psi_s\,\psi_r'\right)\Big/\left(\sum \psi_r^2\right)^2$$

can be seen to be a continous function of x and since the zeros of H_{n-1}, H_n, H_{n+1} are dense in the interval concerned, it follows that (22a) holds for all x in the range under consideration. Thus the Mathematical expectation of the zeros of $\Sigma\, c_n\;\psi_k(x)$ is atleast

$$\frac{1}{\pi\sqrt{3\theta}}\int_{-t_n}^{t_n}\sqrt{2n-x^2}\;\;dx, \quad (t_n = \mu\sqrt{2n})\,.$$

Hence $$EN \geqslant \frac{2n}{\pi\sqrt{3\theta}}\;(\sin^{-1}\mu + \mu\sqrt{1-\mu^2}),\; (\theta > 1,\; \mu < 1)$$

where μ, θ can be near unity showing that the sum $\Sigma\; c_K\;\psi_k(x)$ on an average, oscillate, atleast $\sqrt{1/3}$ 99/100 times the number of oscillations admissible to it by its degree.

§4. For $\alpha > -1$, the Laguerre polynomials $L_k^{(\alpha)}(x)$, (k = 0, 1, 2, .) form an orthogonal set over $(0, \infty)$ with weight function $e^{-x}x^{\alpha}$.

We write the "simple" Laguerre polynomial $L_k^{(0)}(x)$ as $L_k(x)$. In the following lines, we present a brief discussion of the equation $\Sigma\;\; c_k a_k L_k(x) = 0$ which will show that the analysis in the case of general α does not differ substantially from the typical simple case considered here. We begin by observing that $g_k = 1$ for $\alpha = 0$, so that $a_k = 1$. Now to compute $f_n(x)$ write the differential recurrence relations and their consequences as

$$x\,L_n'' = (x-1)\,L_n' - n\,L_n \qquad \text{...(22)}$$

$$x\,L_{n+1}'' = (x-1)\,L_{n+1}' - (n+1)\,L_{n+1} \qquad \text{...(23)}$$

$$x\left(L_{n+1}''\;\;L_n - L_n''\,L_{n+1}\right) = (x-1)\,D_n - L_n\,L_{n+1} \qquad \text{...(24)}$$

$$x\left(L_{n+1}''\;\;L_n' - L_n''\;\;L_{n+1}'\right) = n\,D_n - L_{n+1}\;\;L_n' \qquad \text{...(25)}$$

$$x\left(L_{n+1}'''\;\;L_n - L_n'''\;\;L_{n+1}\right) = (-n+x-3+2/x)\,D_n - L_{n+1}\,L_n' \qquad \text{..(26)}$$

where from $\quad x\,L'_n = x\,L_n + (n+1)\,(L_{n+1} - L_n) \qquad \text{...(27)}$

we have $\quad x\,D_n = x\,(L'_{n+1}\,L_n - L'_n\,L_{n+1}) = -(n+1)\,(L_{n+1} - L_n)^2 - x\,L_n\,L_{n+1}. \qquad \text{...(28)}$

We put $\nu = 4n+2$, $x = \nu\cos^2\theta$ and the oscillatary region $n^{1/3} \leqslant x \leqslant \mu\,(4n+2)$ $(0 < \mu < 1)$ is considered. Here the asymptotic estimate

$$\exp(-x/2)\, L_n(x) = 2\,(-1)^n\, [\pi\,(2n+2)\sin 2\theta]^{-1/2}\,(1+O\,(1/n)).$$

shows that $-x\, D_n \sim (n+1+x)\; L_n^2\,,\; \dfrac{Q_n}{D_n} \sim 1 - \dfrac{1}{x}$

$$\frac{R_n}{D_n} \sim \frac{n-2}{2x}\,,\; \frac{S_n}{D_n} \sim (-n+x-3)/6x$$

so that in the range under consideration,

$$f_n(x) \sim \frac{1}{\pi\sqrt{3}}\sqrt{\frac{n}{x} - \frac{1}{4}}.$$

We integrate $f_n(x)$ on the range $n^{1/3} \leqslant x \leqslant 4n\mu$ to obtain

$$EN_n\,(0, \infty) \geqslant \frac{2n}{\pi\sqrt{3}}\;(\sin^{-1}\mu).$$

If the form of the integrand is any guide, we may hope that for Laguerre polynomials $f_n(x) \sim 1/\pi\sqrt{3}\,\sqrt{n/x-1/4}$ throughout the oscillatory region and say that the actual expectation of the number of real roots is asymptotically equal with $n/\sqrt{3}$. Similarly considerations lead one to hope that for the Hermite polynomials $f_n(x) \sim 1/\pi\sqrt{3}\,\sqrt{2n-x^2}$ throughout the oscillatory region. This will give the same expectation. If this can be done, we shall say that the expectation of real roots of eqn. (1), is asymptotically equal with $n/\sqrt{3}$ when n is sufficiently large in each of the three cases of the sum of normalised classical Harmonic Polynomials.

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