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EXTENSION OF LIAPUNOV THEORY TO THREE POINT BOUNDARY VALUE PROBLEMS

D.R.K.S. Rao, K.N. Murty and M.S.N. Murty

Department of Applied Mathematics, Andhra University, Waltair 530003

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§1. In this paper we shall be concerned with the existence and uniqueness of solutions to three point boundary value problems associated with

$$y''' = f(x, y, y', y'') \qquad \ldots(1.1)$$

where $f(x, y_1, y_2, y_3)$ is a real valued function defined and continuous on $[a, b] \times R^3$. It will be assumed throughout this paper that the solutions of the initial value problems associated with (1.1) exist on $[a, b]$, and are unique. We consider (1.1) alongwith the boundary conditions

$$y(\alpha) = p_1,\ y(\beta) = p_2,\ y(\gamma) = p_3 \qquad \ldots(1.2)$$

where $a \leqslant \alpha < \beta < \gamma \leqslant b$, and $p_1, p_2, p_3 \in R$.

Our approach in this paper is similar to that of Barr and Sherman (1973) and is based on the use of a solution matching technique, and a suitable Liapunov function defined later. Barr and Sherman (1973) proved existence and uniqueness of solutions to three point boundary value problem (1.1) and (1.2) with the following assumption on $f(x, y_1, y_2, y_3)$. The function $f(x, y_1, y_2, y_3)$ is said to satisfy condition A at a point $\beta \in (\alpha, \gamma)$ if there exists α and γ such that $\alpha < \beta < \gamma$;

(i) $y_1 \geqslant z_1,\ y_2 < z_2 \Rightarrow f(x, y_1, y_2, y_3) < f(x, z_1, z_2, z_3)\ \forall x \in (\alpha, \beta]$

and

(ii) $y_1 \leqslant z_1,\ y_2 < z_2 \Rightarrow f(x, y_1, y_2, y_3) < f(x, z_1, z_2, y_3)\ \forall x \in [\beta, \gamma)$.

In this paper we replace condition A by an appropriate Liapunov function and establish existence and uniqueness of solutions to three point boundary value problem (1.1) and (1.2). The results obtained here include more general class of problems than in Barr and Sherman (1973).

Yoshizawa (1966) and Okamura (1942) have demonstrated that the uniqueness of solutions to initial value problems was equivalent to the existence of a Liapunov function. George and Sutton (1970) Barr and Miletta (1975) obtained sufficient conditions for uniqueness of solutions to two point boundary value problems in terms of a Liapunov function. We shall extend the Liapunov theory for the existence and uniqueness of solutions to three point boundary value problem (1.1) and (1.2).

§2. We shall consider the following two point boundary value problems associated with (1.1)

$$y(\alpha) = p_1,\ y(\beta) = p_2,\ y^{(i)}(\beta) = m\ (i = 1, 2) \quad \ldots(2.1_i)$$

and $$y(\beta) = p_2,\ y(\gamma) = p_3\ y^{(i)}(\beta) = m\ (i = 1, 2)\,. \quad \ldots(2.2_i)$$

Suppose y_1 and y_2 be solutions of (1.1) satisfying (2.1_i) or (2.2_i) $(i = 1, 2)$. Write $y = y_1 - y_2$. Then

$$y''' = F(x, y, y', y'') \equiv f(x, y+y_2, y'+y_2', y''+y_2'') - f(x, y_2, y_2', y_2'') \quad \ldots(2.3)$$

$$\text{and } F(x, 0, 0, 0) = 0. \quad \ldots(2.4)$$

Then the boundary conditions (2.1_i) and (2.2_i) respectively become

$$y(\alpha) = 0,\ y(\beta) = 0,\ y^{(i)}(\beta) = 0,\ (i=1, 2) \quad \ldots(2.5_i)$$

$$\text{and } y(\beta) = 0,\ y(\gamma) = 0,\ y^{(i)}(\beta) = 0\ (i = 1, 2). \quad \ldots(2.6_i)$$

Hence $y(x) \equiv 0$ is a solution satisfying (2.3) and (2.5_i) or (2.6_i) $(i = 1, 2)$. Thus we have proved the following:

Lemma 2.1—The problem (1.1) satisfying (2.1_i) and (2.2_i) $(i = 1, 2)$ has a unique solution if and only if $y(x) \equiv 0$ is the only solution of (2.3) satisfying (2.5_i) or (2.6_i) $(i = 1, 2)$.

Definition—A Liapunov function $V(x, y, y', y'')$ is a continuous Locally Lipschitzian real valued function with respect to (y, y', y''). Corresponding to $V(x, y, y', y'')$ we define

$$V_f'(x, y, y', y'') = \liminf_{h\to 0+} \frac{1}{h}\Big[V(x + h, y + hy', y'+hy'', y''+hf) - V(x, y, y'\, y'')\Big]$$

$$V'(x, y, y', y'') = \liminf_{h\to 0+} \frac{1}{h}\Big[V(x + h, y(x+h), y'(x + h), y''(x + h)) - V(x, y, y', y'')\Big]$$

where f is a function defined and continuous on a domain $M = [a, b]\ x\ N$, where $[a, b]$ is a interval on the real line, and $N \subset R^3$.

Lemma 2.2—If $V(x, y, y', y'')$ is a Liapunov function and $y(x)$ is a solution of (1.1) then $V'(x, y, y', y'') = V_f'(x, y, y', y'')$ and $V(x, y, y', y'')$ is non increasing (non decreasing) if and only if $V_f'(x, y, y', y'') \leqslant 0$ $(V_f'(x, y, y', y'') \geqslant 0)$.

PROOF: Analogous to the proof of Yoshizawa (1966, p. 4).

Lemma 2.3—For F defined in (2.3), if there exists a Liapunov function $V(x, y, y', y'')$ defined on M such that

(i) $V(x, y, y', y'') = 0$ if $y = 0$

(ii) $V(x, y, y', y'') > 0$ if $y \neq 0$

(iii) $V_F'(x, y, y', y'') \geqslant 0$ in the interior of M,

then for each $m \in R$ there exists atmost one solution to the following two point boundary value problems ; (1.1) satisfying (2.1_i) $(i = 1, 2)$.

PROOF : The proof of the problem (1.1) satisfying (2.1_2) will be given, similar proof holds for the other boundary value problem. Suppose $y_1(x)$ and $y_2(x)$ are solutions of (1.1) satisfying (2.1_2). Write $W(x) = y_1(x)-y_2(x)$. Then

$$W''' = F(x, W, W', W''),\ W(\alpha) = 0,\ W(\beta) = 0,\ W''(\beta) = 0 \qquad \ldots(2.7)$$

where $$F(x, 0, 0, 0) = 0.$$

From Lemma 2.1 it sufficies to show $W(x) \equiv 0$ is the only solution of (2.7). Suppose $\phi(x)$ is a non trivial solution of (2.7) then there exists an $\eta \in (\alpha, \beta)$ such that $\phi(\eta) \neq 0$. Hence

$$V(\eta, \phi(\eta), \phi'(\eta), \phi''(\eta)) > 0. \qquad \ldots(2.8)$$

Since $V'_F(x, y, y', y'') \geqslant 0$ in the interior of M, and from Lemma 2.2 it follows that V is non-decreasing along the solution $\phi(x)$. Thus $\eta < \beta$ implies

$$V(\eta, \phi(\eta), \phi'(\eta), \phi''(\eta)) \leqslant V(\beta, \phi(\beta), \phi'(\beta), \phi''(\beta)), \leqslant 0. \qquad \ldots(2.9)$$

Hence (2.8) and (2.9) contradict each other. Hence $y_1(x) \equiv y_2(x)$.

Lemma 2.4—For F defined in (2·3) if there exists a Liapunov function defined on M as in Lemma (2.3), then for each $m \in R$ there exists atmost one solution of (1.1) satisfying (2.2_i) $(i = 1, 2)$.

Proof: Analogous to the proof in Lemma 2.3.

§3. In this section we prove existence and uniqueness of solutions to three point boundary value problems associated with (2.1) by using Liapunov function defined in Lemma 2.3.

Theorem 3.1—Let $p_1, p_2, p_3 \in R$, $a < \beta < b$ and $a \leqslant \alpha < \beta < \gamma \leqslant b$. Suppose that for each $m \in R$.

(i) there exist solutions of (1.1) satisfying (2.1_i) and (2.2_i) $(i = 1, 2)$

(ii) $V(x, y, y', y'')$ is a Liapunov function as in Lemma 2.3.

Then there exists a unique solution to the boundary value problem (1.1) satisfying (1.2).

Proof : Let $y_1(x, m)$ denote the solution of (1.1) satisfying (2.1_2) with second derivative m at $x = \beta$. By hypothesis, if $m_2 > m_1$

$$y_1''(x, m_2) > y_1''(x, m_1).$$

From Lemmas (2.3) and (2.4) the solutions of (1.1) satisfying (2.1_i) and (2.2_i) $(i=1.2)$ are unique.

Since $y_1''(x, m_1)$ and $y_1''(x, m_2)$ are continuous functions, either

(i) $y_1''(x, m_2) > y_1''(x, m_1)\ \forall\ x \in [\alpha, \beta]$

(ii) $y_1''(\alpha, m_2) = y_1''(\alpha, m_1)$ and $y_1''(x, m_2) > y_1''(x, m_1)\ \forall\ x \in (\alpha, \beta]$.

or

(iii) $\exists\ \delta \in (\alpha, \beta)$ such that

$$y_1''(\delta, m_2) = y_1''(\delta, m_1) \text{ and } y_1''(x, m_2) > y_1''(x, m_1)\ \forall\ x \in (\delta, \beta] \text{ holds.}$$

Suppose (i) or (ii) holds. Define $W(x) = y_1(x, m_2) - y_1(x_1, m_1)$.
$W(\alpha) = 0$, $W(\beta) = 0 \Rightarrow$ there exists $n \in (\alpha, \beta)$ such that

$$y_1'(n, m_2) = y_1'(n, m_1). \quad \ldots(3.1)$$

W satisfies $W''' = F(x, W, W', W'')$.

Since (i) or (ii) holds and (3.1) implies $y_1'(x, m_2) > y_1'(x, m_1)\ \forall x \in (n, \beta]$.

Suppose (iii) holds. Claim. There exists a point $q \in [\delta, \beta)$ such that $y_1'(q, m_2) \geqslant y_1'(q, m_1)$ and from (iii) follows $y_1'(x, m_2) > y_1'(x, m_1) \forall x \in (q, \beta]$.

Suppose to the contrary $y_1'(x, m_2) < y_1'(x, m_1)\ \forall\ x \in [\delta, \beta)$. ...(3.2)

Since $y_1(\beta, m_2) = y_1(\beta, m_1)$ implies $W(\delta) > 0$.
Hence $V(\delta, W(\delta)), W'(\delta), W''(\delta) > 0$. ...(3.3)

Since $V_F'(x, y, y', y'') \geqslant 0$ in the interior of M, and from Lemma 2.2 it follows that $V(x, y, y', y'')$ is non decreasing along $W(x)$.
Since $\delta < \beta$ implies

$$V(\delta, W(\delta), W'(\delta), W''(\delta)) \leqslant V(\beta, W(\beta), W'(\beta\delta), W''(\beta)) \leqslant 0. \quad \ldots(3.4)$$

Thus (3.3) and (3.4) contradict each other. Hence $y_1'(x, m_2) > y_1'(x, m_1)\ \forall\ x \in (q, \beta]$.

Thus in all cases $y_1'(\beta, m)$ considered as a function of m is strictly increasing. Let $y_2(x, m)$ denote the solution of (1.1) satisfying (2.2_2). A proof similar to above shows that $y_2'(\beta, m)$ considered as a function of m is strictly decreasing.

Now $y_1'(\beta, .) : R \longrightarrow R$. Claim. $y_1'(\beta, .) : R \xrightarrow{\text{on to}} R$.

Let $z_0 \in R$. The boundary value problem

$$y''' = f(x, y, y', y''),\ y(\alpha) = p_1,\ y(\beta) = p_2,\ y'(\beta) = z_0$$

has a unique solution ϕ. Let $\phi''(\beta) = p$. Hence ϕ is a solution of (1.1) satisfying (2.1_2). But this problem has a unique solution. Hence our claim is true.

Similarly $y_2'(\beta, \cdot) : R \xrightarrow{\text{on to}} R$. Thus both $y_1'(\beta, m)$ and $y_2'(\beta, m)$ are continuous strictly monotonic functions of m whose ranges are the set of all real numbers.

Denote $Y'(\beta, m) = y_1'(\beta, m) - y_2'(\beta, m)$

$$Y'(\beta, m) \longrightarrow \infty \text{ as } m \longrightarrow \infty,\ Y'(\beta, m) \longrightarrow -\infty \text{ as } m \longrightarrow -\infty.$$

Thus there exists an m_0 such that $y_1'(\beta, m_0) = y_2'(\beta, m_0)$.

According to hypothesis $y_1(\beta, m_0) = y_2(\beta, m_0)$ and $y_1''(\beta, m_0) = y_2''(\beta, m_0)$.

$$\text{Thus } y(x) = \begin{cases} y_1(x, m_0) & \alpha \leqslant x \leqslant \beta \\ y_2(x, m_0) & \beta \leqslant x \leqslant \gamma \end{cases}$$

is a solution of (1.1) satisfying (1.2). Suppose ϕ_1 and ϕ_2 are two distinct solutions of (1.1) satisfying (1.2). Write $\phi = \phi_1 - \phi_2$

$$\phi''' = F(x, \phi, \phi', \phi''), \phi(\alpha) = 0, \phi(\beta) = 0, \phi(\gamma) = 0. \quad ...(3.5)$$

where $F(x, 0, 0, 0) = 0$. Claim: $\phi(x) \equiv 0 \ \forall \ x \in [\alpha, \gamma]$.
Suppose $\phi_0(x)$ is a non-trivial solution of (3.5).
Since $\phi_0(\alpha) = 0, \phi_0(\gamma) = 0 \Rightarrow$ there exists a point $s \in (\alpha, \gamma)$ such that $\phi_0(s) \neq 0$.

Hence $V(s, \phi_0(s), \phi_0'(s), \phi_0''(s)) > 0. \quad ...(3.6)$

Since $V_F'(x, y, y', y'') \geqslant 0$ and from Lemma 2.2 it follows that V is non-decreasing along the solution $\phi_0(x)$.

Since $s < \gamma$, $V(s, \phi_0(s), \phi_0'(s), \phi_0''(s)) \leqslant V(\gamma, \phi_0(\gamma), \phi_0'(\gamma), \phi_0''(\gamma)) \leqslant 0. \quad ...(3.7)$

Thus (3.7) and (3.8) contradict each other. Hence $\phi_1(x) = \phi_2(x) \ \forall \ x \in [\alpha, \gamma]$.

The following example shows the existence of a Liapunov funetion as in Lemma 2.3.

$V(x, y, y', y'') = y^2$ is a Liapunov function for $y''' = 0$ on $M = [-1, 1] \times N$, where $N = [(y, y', y'') \in R^3$ such that $yy' = 0$ and $y' y'' = 0]$.

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