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FIXED POINT THEOREMS ON CONTRACTIVE MAPPINGS

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In the present paper we prove fixed point theorem for mapping in a normed space satisfying a certain contractive definition, introduced by Pachpatte.

More recently Pachpatte (1981) proved some unique fixed point theorems for a mapping T of a metric space (X, d) into itself satisfying,

$$d(Tx, Ty) \leqslant q \max \left\{ d(x,y), \frac{d(y, Ty)\,[1+d(x, Tx)]}{1+d(x, Tx)}, \frac{1}{2}\,\frac{d(x, Ty)\,[1+d(x, Tx)+d(y, Ty)]}{1+d(x, y)} \right\}, \qquad \ldots(1)$$

for all x, y in X where $0<q<1$ and generalized the Banach contraction principle (Liusternik and Sobolev 1961, p. 27).

Rhoades (1974) has shown that "For a mapping T satisfying a certain contractive definitions, if the sequence of Mann iterates converges, then it converges to a fixed point of Ty. For related literature see Rhoades (1977), where the author has arranged a bulk of mappings satisfying contractive type definitions studied by different authors.

Let T be a self mapping of a Banach space X. The Mann iterative process associated by T is defined in the following manner:

Let x_0 be in X and set $x_{n+1} = (1-c_n)\, x_n + c_n T x_n$, for $n>0$.

Where $\{c_n\}$ satisfies (*i*) $c_0 = 1$, (ii) $0 < c_n < 1$ for $n > 0$, (iii) $\Sigma\, c_n$ diverges. In this paper we impose an additional restriction on $\{c_n\}$, namely (iv) $\lim_n c_n = h > 0$.

In the present paper we prove some fixed point theorems using the technique as appeared in Rhoades (1974) for mapping T of X satisfying contractive definition (1).

Theorem 1—Let X be a closed convex subset of a normed space. T a self mapping of X satisfying (1) on X, $\{x_n\}$ the sequence of Mann iterates associated with T be same as defined above, where $\{c_n\}$ satisfies (i), (ii), (iv). If $\{x_n\}$ converges in X, then it converges to a fixed point of T.

PROOF : Let $z \in X$ satisfy $\lim_{n\to\infty} x_n = z$. Then

$$d(z, Tz) \leqslant d(z, x_{n+1}) + d(x_{n+1}, Tz),$$
$$\leqslant d(z, x_{n+1}) + \| (1-c_n) x_n + c_n T x_n - Tz \|,$$
$$\leqslant d(z, x_{n+1}) + \| (1-c_n) x_n - (1-c_n) Tz + c_n T x_n - c_n Tz \|,$$
$$\leqslant d(z, x_{n+1}) + (1-c_n) d(x_n, Tz) + c_n d(Tx_n, Tz),$$
$$\leqslant d(z, x_{n+1}) + (1-c_n) d(x_n, Tz) +$$
$$c_n q \max \left\{ d(x_n, z), \frac{d(z, Tz)[1 + d(x_n, Tx_n)]}{1 + d(x_n, z)}, \frac{1}{2} \frac{d(x_n, Tz)[1 + d(x_n, Tx_n) + d(z, Tx_n)]}{1 + d(x_n, z)} \right\},$$

we note that $d(x_n, Tx_n) = d(x_n, x_{n+1})/c_n$, and

$$d(z, Tx_n) \leqslant d(z, x_n) + d(x_n, Tx_n), \leqslant d(z, x_n) + d(x_n, x_{n+1})/c_n,$$

therefore the above inequality reduces to,

$$d(z, Tz) \leqslant d(z, x_{n+1}) + (1-c_n) d(x_n, Tz) +$$
$$c_n q \max \left\{ d(x_n, z), \frac{d(z, Tz)[1 + d(x_n, x_{n+1})/c_n]}{1 + d(x_n, z)}, \frac{1}{2} \frac{d(x_n, Tz)[1 + 2d(x_n, x_{n+1})/c_n + d(z, x_n)]}{1 + d(x_n, z)} \right\}.$$

Now taking the limit n tends to infinity, and using (iv), we have,

$$d(z, Tz) \leqslant (1 - h + qh) d(z, Tz),$$

which leads to result $d(z, Tz) = 0$ i.e. z is the fixed point of T.

We extend the above result for a pair of mappings. We have the following theorem.

Theorem 2—Let X be a closed convex subset of a normed space and and let T_1 and T_2 be two self mappings of X satisfying

$$d(T_1 x, T_2 y) \leqslant q \max \left\{ d(x, y), \frac{d(y, T_2 y)[1 + d(x, T_1 x)]}{1 + d(x, y)}, \frac{1}{2} \frac{d(x, T_2 y)[1 + d(x, T_1 x) + d(y, T_1 x)]}{1 + d(x, y)} \right\}. \quad \ldots(2)$$

For all x, y in X where $0 < q < 1$.

Let the sequence $\{x_n\}$ be defined in accordance with the Mann iterates associated with T_1 and T_2 as given below:

For $x_0 \in X$, set $x_{2n+1} = (1-c_n) x_{2n} + c_n T_1 x_{2n}$, and $x_{2(n+1)} = (1-c_n) x_{2n+1} + c_n T_2 x_{2n+1}$, for $n = 0, 1, 2 \ldots$ where $\{c_n\}$ satisfies (i), (ii), and (iv).

If $\{x_n\}$ converges to z in X, then z is the common fixed point of T_1 and T_2.

PROOF : Let z be in X such that $\lim_{n \to \infty} x_n = z$.

We shall show that z is the common fixed point of T_1 and T_2. Consider,

$$d(z, T_2 z) \leqslant d(z, x_{2n+1}) + \| (1-c_n) x_{2n} + c_n T_1 x_{2n} - T_2 z \|,$$
$$\leqslant d(z, x_{2n+1}) + \| (1-c_n) x_{2n} - (1-c_n) T_2 z + c_n T_1 x_{2n} - c_n T_2 z \|,$$
$$\leqslant d(z, x_{2n+1}) + (1-c_n) d(x_{2n}, T_2 z) + c_n d(T_1 x_{2n}, T_2 z),$$
$$\leqslant d(z, x_{2n+1}) + (1-c_n) d(x_{2n}, T_2 z) +$$

(*equation continued on p.* 428)

$$c_n\, q \max \left\{ d(x_{2n}, z), \frac{d(z, T_2 z)\,[1 + d(x_{2n}, T_1 x_{2n})]}{(1 + d(x_{2n}, z)}, \right.$$

$$\left. \frac{1}{2} \frac{d(x_{2n}, T_2 z)\,[1 + d(x_{2n}, T_1 x_{2n}) + d(z, T_1 x_{2n})]}{1 + d(x_{2n}, z)} \right\}$$

$$\leqslant d(z, x_{2n+1}) + (1 - c_n).\, d(x_{2n}, T_2 z) +$$

$$c_n\, q \max \left\{ d(x_{2n}, z), \frac{d(z, T_2 z)\,[1 + d(x_{2n}, x_{n2+1})/c_n]}{1 + d(x_{2n}, z)}, \right.$$

$$\left. \frac{1}{2} \frac{d(x_{2n}, T_2 z)\,[1 + 2d(x_{2n}, x_{2n+1})/c_n + d(z, x_{n2})]}{1 + d(x_{2n}, z)} \right\}$$

letting n tend to infinity, we obtain using (iv) also,

$$d(z, T_2 z) \leqslant (1 - h + hq)\, d(z, T_2 z),$$

which implies $d(z, T_2 z) = 0$. Similarly one can easily show that $d(z, T_1 z) = 0$. Hence z is the common fixed point of T_1 and T_2.

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