Indian J. pure appl. Math., 13(4) : 429—432, April 1982

SOME RANDOM FIXED POINT THEOREMS FOR MULTIVALUED AND SINGLE-VALUED MAPPINGS

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(*Received* 3 *March* 1981; *after revision* 9 *November* 1981)

In the present paper two results on random fixed points are given. First result generalizes the result of Itoh (1977) for random multivalued contraction mapping. The second result is a random version of a result of Jungck (1976) for commuting nonlinear mappings.

§1. A random version of a fixed point theorem for multivalued contraction mapping of Nadler (1969) was given by Itoh (1977). In this paper we give two results on random fixed points. Our first result is concerned with random fixed points of multivalued generalized contraction type of mapping [for example see Ray (1976)] on a Polish Space. Second result is a random version of a result of Jungck (1976) for commuting nonlinear mappings on a Polish space.

§2. Throughout this paper (X, d) will denote a Polish Space, i.e., a separable complete metric space, and (T, A) a measurable space. We denote $d(x,B) = \inf \{d(x, y): y\in B\}$ for any $x\in X$ and $B\subset X$. Let 2^X be the family of all subsets of X*, $CB(X)$ the family of all closed and bounded subsets of X and B the σ-algebra of Borel subsets of X, respectively. Let H be the Hausdorff metric on $CB(X)$ induced by d. A mapping $F: T\to 2^X$ is called (A—) measurable if for any open subset B of X, $F^{-1}(B)\in A$, where $F^{-1}(B) = \{ t\in T : F(t) \cap B \neq \phi\}$. A mapping $u : T\to X$ is said to be a measurable selector of a measurable mapping $F : T\to 2^X$ if u is measurable and for any $t\in T$, $u(t)\in F(t)$. A mapping $F : X\to 2^X$ is called k-Lipschitz, where $k\geqslant 0$, if for every $x, y\in X$, $H(F(x), F(y)) \leqslant k\, d(x, y)$. If $k < 1$ then F is called a contraction.

§3. In this section we state some results on measurability obtained by Itoh (1977). For proofs we refer the reader to the same reference.

Proposition 3.1—Let $\{F_n\}$ be a sequence of measurable mappings $F_n:T\to CB(X)$. and $F : T\to CB(X)$ a mapping such that for each $t\in T$, $H(F_n(t), F(t))\to 0$ as $n\to\infty$. Then F is measurable.

Proposition 3.2—Let $F : T\times X\to CB(X)$ be a mapping such that for each $x\in X$, $F(., x)$ is measurable and for each $t\in T$, $F(t, .)$ is $k(t)$—Lipschitz. Let $u:T\to X$ be a

*Here we follow the convention as in Itoh (1977).

measurable mapping, then the mapping $G:T\to CB(X)$ defined by $G(t) = F(t, u(t))$ $(t\in T)$ is measurable.

Proposition 3.3—Let Y be a metric space, $f: T\times X\to Y$ a mapping such that for any $t\in T, f(t, .)$ is continuous and for any $x\in X, f(., x)$ is measurable. Let $F:T\to 2^X$ be a measurable mapping such that for each $t\in T$, $F(t)$ is nonempty closed, and U an open subset of Y. Then the mapping $G:T\to 2^X$ defined by $G(t) = \{x\in F(t) : f(t, x) \in U\}$ $(t\in T)$ is measurable.

Proposition 3.4—Let $F, G:T\to CB(X)$ be measurable mappings, $u:T\to X$ a measurable selector of F, $r:T\to (0, \infty)$ a measurable function. Then there exists a measurable selector $v:T\to X$ of G such that for any $t\in T$, $d(u(t), v(t)) \leqslant H(F(t), G(t) + r(t))$.

§4. We state and prove our first result.

Proposition 4.1—Let $F:T\times X\to CB(X)$ be a mapping such that for each $x\in X$, $F(., x)$ is measurable and for each $t\in T$, $F(t, .)$ is continuous. Let F satisfy a generalized contraction type of condition as given below.

$$H(F(t, x), F(t, y)) \leqslant \alpha(t)\, d(x, F(t, x)) + \beta(t)\, d(y, F(t, y)) + \gamma(t)\, d(x, y) \quad \text{for all } x, y\in X, \qquad \ldots(*)$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are measurable maps from T into $[0, 1)$ with the property that $\alpha(t) + \beta(t) + \gamma(t) < 1$.

Then there exists a measurable mapping $u : T \to X$ such that for any $t \in T$, $u(t) \in F(t, u(t))$.

PROOF: Let $A_1 = \{t \in T : \alpha(t) > 0\}$, $A_2 = \{t \in T : \beta(t) > 0\}$ and $A_3 = \{t \in T : \gamma(t) > 0\}$.

Put $E = A_1 \cap A_2 \cap A_3$ and $S = T\setminus E$., then E and $S\in A$.

First we consider on the set E. Take a measurable mapping $v_0 : E\to X$.

By proposition 3.2, the mapping $F(., v_0(.)) : E \to CB(X)$ is measurable, hence there exists a measurable selector $v_1 : S \to X$ of $F(., v_0(.))$ by Kuratowski and Ryll-Nardzewski (Itoh 1977). Then by proposition 3.4, there exists a measurable selector $v_2 : E \to X$ of $F(., v_1(.))$ such that for any $t \in T$;

$$d(v_1(t), v_2(t)) \leqslant H(F(t, v_0(t)), F(t, v_1(t))) + \frac{\alpha(t) + \gamma(t)}{1-\beta(t)}$$

(choosing $r(t) = (\alpha(t)+\gamma(t))/(1-\beta(t))$).

By proposition 3.4 again, there exists a measurable Selector $v_3 : E \to X$ of $F(., v_2(.))$ such that for $t \in T$,

$$d(v_2(t), v_3(t)) \leqslant H(F(t, v_1(t)), F(t, v_2(t))) + \left(\frac{\alpha(t)+\gamma(t)}{1\ \ \beta(t)}\right)^2.$$

By induction, we can choose a sequence of measurable mappings $v_n : E \to X$ such that for each $t\in T$, $v_n(t) \in F(t, v_{n-1}(t))$ and

$$d(v_n(t), v_{n+1}(t)) \leqslant H(F(t, v_{n-1}(t)), F(t, v_n(t))) + \left(\frac{\alpha(t)+\gamma(t)}{1-\beta(t)}\right)^n. \qquad (n = 1,2,\ldots\ldots)$$

Applying condition (*) we now have

$$d(v_n(t), v_{n+1}(t)) \leqslant \alpha(t) \; d(v_{n-1}(t), \; F(t, v_{n-1}(t))) + \beta(t)\, d(v_n(t), F(t, v_n(t))) + \gamma(t)\, d(v_n(t), v_{n-1}(t)) + \left(\frac{\alpha(t)+\gamma(t)}{1-\beta(t)}\right)^n$$

$$\leqslant \alpha(t)\, d(v_{n-1}(t), v_n(t)) + \beta(t)\, d(v_n(t), v_{n+1}(t)) + \gamma(t)\, d(v_{n-1}(t), v_n(t)) + \left(\frac{\alpha(t)+\gamma(t)}{1-\beta(t)}\right)$$

i.e., $d(v_n(t), v_{n+1}(t)) \leqslant \dfrac{\alpha(t)+\gamma(t)}{1-\beta(t)} \; d(v_{n-1}(t), v_n(t)) + \dfrac{(\alpha(t)+\gamma(t))^n}{(1-\beta(t))^{n+1}}$.

But then

$$d(v_n(t), v_{n+1}(t)) \leqslant \sum_{i=n}^{n+k-1} d(v_i(t), \; v_{i+1}(t))$$

$$\leqslant \sum_{j=n}^{n+k-1} \lambda^j \, d(v_0(t), v_1(t)) + \frac{1}{\alpha(t)+\gamma(t)} \sum_{j=n}^{n+k-1} j\,\lambda^{j+1}$$

where $\lambda = \dfrac{\alpha(t)+\gamma(t)}{1-\beta(t)} < 1$, for all $t \in T$ $(k, n > 1)$.

From which by taking limit as $n \to \infty$ it follows that $\{v_n(t)\}$ is a cauchy sequence in X. hence converges to some $v(t) \in X$. Now it follows that for any n

$$\begin{aligned} d(v(t), F(t, v(t)) &\leqslant d(v(t))\, v_n(t)) + d(v_n(t), F(t, v(t)) \\ &\leqslant d(v(t), v_n(t), + H(F(t, v_{n-1}(t)), F(t, v(t))) \\ &\leqslant d(v(t)), v_n(t) + \alpha(t)\, d(v_{n-1}(t), F(t, v_{n-1}(t))) \\ &\quad + \beta(t)\, d(v(t), F(t, v(t))) + \gamma(t)\, d(v_{n-1}(t), v(t)). \end{aligned}$$

The right-hand side of above inequality converges to $(\alpha(t) + \beta(t))\, d(v(t), F(t, v(t))$ by continuity of $F(t,.)$ as $n \to \infty$. Since $\alpha(t) + \beta(t) < 1$ above inequality gives $d(v(t), F(t, v(t))) = 0$, i.e., $v(t) \in F(t, v(t))$, since $F(t, v(t))$ is closed. The mapping $v(t)$ is the pointwise limit of measurable mappings $v_n(t)$, hence it is measurable.

In case $t \in S$, i.e., $t \in A'_1 \cup A'_2 \cup A'_3$ (where the prime denotes complements) and $t \notin A_1 \cap A_2 \cap A_3$ then by following the same analysis as above we can get a function, say by the same name as $v(t) : S \to X$ which is measurable and for $t \in S$, $t \notin A_1 \cap A_2 \cap A_3$, $v(t) \in F(t, v(t))$. For $t \in A'_1 \cap A'_2 \cap A'_3$ we have $H(F(t, x), F(t, y)) = 0$. Thus we can set $F(t, x) = F_0(t)$ for each $t \in A'_1 \cap A'_2 \cap A'_3$ where $F_0 : A'_1 \cap A'_2 \cap A'_3 \to CB(X)$ is measurable. By Kuratowski-Ryll-Nardzewski (Itoh 1977), there exists a measurable selector (denoted by the same name) $v : A'_1 \cap A'_2 \cap A'_3 \to X$ of F_0 such that $v(t) \in F(t, v(t))$ for $t \in A'_1 \cap A'_2 \cap A'_3$. Which Completes the proof.

§ 5. In this section we give a random version of a result of Jungck (1976). If $f : T \times X \to X$ and $g : T \times X \to X$ are two mappings such that $f(t, g(t, x_0)) = g(t, f(t, x_0)$ for each $x_0 \in X$ then we say that f and g satisfy property (c). We state our proposition as follows.

Proposition 5.1—Let f and g be two mappings from $T \times X$ into X. For each $t \in T$ let $f(t,.)$ be continuous and for each $x \in X$; let $f(.,x)$ and $g(.,x)$ be measurable. Let f and g satisfy property (c)...Also we assume that for each $t \in T$, $g(t, X) \subset f(t, X)$. If f and g satisfy the following condition.

$d(g(t, x), g(t, y)) \leqslant \alpha(t)\, d(f(t, x), f(t, y))$——(*)

for $x, y \in X$ and each $t \in T$, with $0 < \alpha(t) < 1$ and $\alpha(t)$ measurable then $f(t,.)$ and $g(t,.)$ have a fixed point in X given by $g(t, s(t))$ which is measurable in t.

PROOF: Let $x_0 \in X$. Since $g(t, X) \subset f(t, X)$, there is an x_1 such that $f(t, x_1) = g(t, x_0)$. Similarly there is an $x_2 \in X$ such that $f(t, x_2) = g(t, x_1)$. In this process we define x_n such that $f(t, x_{n+1}) = g(t, x_n)$.

Let $y_n(t) = f(t, x_n)$, then by a method similar to Jungck (1976) it can be shown that

$$d(y_n(t), Y_{n+1}(t)) \leqslant \beta(t)\, d(y_n(t), y_{n-1}(t)) \quad \ldots(2)$$

where $\beta(t)$ is measurable and $0 < \beta(t) < 1$.

$\{y_n(t)\}$ is a sequence of measurable functions from $T \to X$.

From Lemma 1 of Jungck (1976) it follows that $\{y_n(t)\}$ is a Cauchy sequence in X, hence it converges to some $s(t) \in X$. Since $f(t,.)$ is continuous, hence it follows that $f(t, y_n(t)) \to f(t, s(t))$.

By Proposition 3.2 (for a particular case when F is single-valued) $f(t, y_n(t))$ is measurable for each n. Hence $f(t, s(t))$ being the pointwise limit of a sequence of measurable functions is measurable in t. Also $g(t, y_n(t)) = g(t, f(t, x_n)) = f(t, g(t, x_n)) = f(t, f(t, x_{n+1})) = f(t, y_{n+1}(t)) \to f(t, s(t))$ as $n \to \infty$, since $f(t,.)$ is continuous for each t. From which applying condition (*) we can see that $g(t, s(t)) = f(t, s(t))$.

Now $d(g(t, y_n(t)), g(t, g(t, s(t))) \leqslant \alpha(t)\, d(f(t, y_n(t)), f(t, g(t, s(t)))$

$= \alpha(t)\, d(f(t, y_n(t)), g(t, f(t, s(t))))$

$= \alpha(t)\, d(f(t, y_n(t)), g(t, g(t, s(t)))$

as $n \to \infty$, we have

$$d(g(t, s(t)), g(t, g(t, s(t)))) \leqslant \alpha(t)\, d(g(t, s(t)), g(t, g(t, s(t))))$$

i.e., $(1-\alpha(t))\, d(g(t, s(t)), g(t, g(t, s(t)))) = 0$

i.e., $g(t, s(t)) = g(t, g(t, s(t)))$

hence $g(t, s(t))$ is a fixed point of $f(t,.)$ and $g(t,.)$, which is also measurable in t.

ADDED IN THE PROOF—*Remark*: The separability assumption on X in § 2 is needed, in the proofs of subsequent propositions (Prop. 3.2, 3.3 and 3.4).

ACKNOWLEDGEMENT

The author expresses his sincere thanks to the referee for his valuable suggestions to improve over the earlier version of the paper.

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