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DIFICIENCY INDICES OF SECOND-ORDER MATRIX DIFFERENTIAL OPERATORS

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We consider the second-order matrix differential operator N given by

$$N \equiv \begin{pmatrix} -\frac{d^2}{dx^2} + p(x) & r(x) \\ r(x) & -\frac{d^2}{dx^2} + q(x) \end{pmatrix}$$

and discuss sufficient conditions on $p(x)$, $q(x)$ and $r(x)$ for N not to be limit-2, and to be limit-3. The results have been extended to the case in which $p(x)$, $q(x)$ and $r(x)$ are oscillating. A theorem for N to be limit-2 has also been proved.

§1. Let N denote the matrix differential operator

$$N \equiv \begin{pmatrix} -\frac{d^2}{dx^2} + p(x) & r(x) \\ r(x) & -\frac{d^2}{dx^2} + q(x) \end{pmatrix} \quad \ldots(1.1)$$

and ϕ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix $\phi = \left(\frac{u}{v}\right)$. Here $p(x)$, $q(x)$ and $r(x)$ are real-valued functions defined in $[0, \infty)$ such that $p(x)$, $q(x)$ and $r(x)$ are $L[0, X]$ for all $X > 0$.

Bhagat (1969b) has shown that, for each λ such im $\lambda \neq 0$, the matrix differential equation

$$(N-\lambda)\,\phi = 0 \;(0 \leqslant x < \infty) \quad \ldots(1.2)$$

has at least two linearly independent solutions which are $L^2[0, \infty)$.

N is said to be limit-2, limit-3 or limit-4 according as the equation (1.2) has two, three or four linearly independent solutions belonging to $L^2[0, \infty)$. For similar classification of fourth-order differential operater associated with

$$\frac{d^4}{dx^4} - \frac{d}{dx}\, p(x)\, \frac{d}{dx} + q(x) \quad \ldots(1.3)$$

one is referred to Everitt (1968, 1969) and Eastham (1971, 1974).

Shaw and Bhagat (1974) and Gadamsi and Mahto (1978) have considered sufficient conditions on p, q and r for N to be limit-2, and Bhagat (1979) for N not to be limit-2. In the present paper we have taken different forms of $p(x)$, $q(x)$ and $r(x)$ and have discussed conditions on them for N to be limit-3 and not to be limit-2. A limit-2 case has also been discussed. We shall use the notations of Bhagat (1969a, b).

§2. Let D denote the set of complex-valued vectors $\phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ such that

(1) $\phi(x)$ is $L^2[0, \infty)$,

(2) $\phi'(x)$ is absolutely continuous in $[0, X]$ for all $X>0$.

(3) $N\phi(x)$ is $L^2\,[0, \infty)$.

It is known that (see Shaw and Bhagat (1974)) N is limit-2 if and only if $[\phi_1\, \phi_2]\,(x) \to 0$ as $x \to \infty$ for all $\phi_1(x)$ and $\phi_2(x)$ in D, where $[\phi_1\, \phi_2]\,(x)$ is the bilinear concomitant of ϕ_1 and ϕ_2 defined as (see Bhagat 1969a)

$$\Big[\phi_1\, \phi_2\Big](x) = u_1^{(1)}\, \bar{u}_2 - u_1 \bar{u}_2^{(1)} + v_1^{(1)} \bar{v}_2 - \bar{v}_1 v_2^{(1)}. \qquad \text{..(2.1)}$$

§3. Now we shall prove the following theorem.

Theorem—In some interval $[X, \infty)$ $(X>0)$, let

$$p(x) = -ax^{\alpha} + S_1^{(2)}\,(x) + x^{\alpha/4}h_1(x),$$

$$q(x) = -bx^{\beta} + S_2^{(2)}\,(x) + x^{\beta/4}h_2(x),$$

$$r(x) = O\,(1) \text{ or } r(x) \begin{pmatrix} x^{-\alpha/4} \\ x^{-\beta/4} \end{pmatrix} \text{ is } L^2\,[X, \infty)$$

where

(i) a, b, α, β are real constants ; $a, b > 0$ and $\alpha, \beta > 2$;

(ii) $S_1(x)$ and $S_2(x)$ are real valued with $S_1(x) \to 0$, $S_2(x) \to 0$ as $x \to \infty$;

(iii) the vector $\begin{pmatrix} S_1(x) \\ S_2(x) \end{pmatrix}$ is twice differentiable in $[X, \infty)$ and

$$\begin{pmatrix} x^{3\alpha/4}\, S_1 \\ x^{3\beta/4}\, S_2 \end{pmatrix}, \begin{pmatrix} x^{-\alpha/4-1}\;\; S_1^{(1)} \\ x^{-\beta/4-1}\;\; S_2^{(1)} \end{pmatrix} \text{ and } \begin{pmatrix} x^{-\alpha/4}\, S_1\, S_1^{(2)} \\ x^{-\beta/4}\, S_2\, S_2^{(2)} \end{pmatrix} \text{ are } L^2[X, \infty) \text{ ;}$$

(iv) the vector $\begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}$ is real-valued and $L^2\,[X, \infty)$.

In $[0, X]$, let $p(x)$ and $q(x)$ and $r(x)$ be $L^2[0, X]$. Then N is not limit-2.

If $r(x) = O\,(1)$ and in addition, at least one of $p(x)$ and $q(x)$ is non-negative in $[X, \infty]$, then N is limit-3.

In order to prove that N is not limit-2 it is sufficient to prove that there are vectors ϕ_1 and ϕ_2 in D such that

$$[\phi_1\, \phi_2]\,(x) \to l, \quad \text{where } l \neq 0, \text{ as } x \to \infty. \qquad \text{...(3.1)}$$

Let us take

$$\phi_1 = \phi_2 = \begin{bmatrix} F(x) \exp\Big\{ i \int_0^x H(t)\, dt \Big\} \\ \\ G(x) \exp\Big\{ i \int_0^x J(t)\, dt \Big\} \end{bmatrix}, \qquad \text{...(3.2)}$$

where $F(x)$, $G(x)$, $H(x)$ and $J(x)$ are real-valued in $[0, \infty)$, $F^{(2)}(x)$, $G^{(2)}(x)$, $H^{(1)}(x)$ and $J^{(1)}(x)$ exist and are continuous. We have to determine $F(x)$, $G(x)$, $H(x)$ and $J(x)$ so that (3.1) hold.

Substituting from (3.2) in (2.1) we get

$$[\phi_1\, \phi_1](x) = -2i(HF^2 + JG^2). \qquad \ldots(3.3)$$

Calculating for $N\phi_1$, we get

$$N\phi_1 = \begin{bmatrix} \left\{(H^2+p)F - F^{(2)} - i\,(2HF^{(1)} + H^{(1)}F)\right\} \exp\left\{ i \int_0^x H(t)\,dt \right\} + rG \exp\left\{ i \int_0^x J(t)\,dt \right\} \\ \left\{ (J^2+q)G - G^{(2)} - i\,(2JG^{(1)} + J^{(1)}G) \right\} \exp\left\{ i \int_0^x J(t)\,dt \right\} + rF \exp\left\{ i \int_0^x H(t)\,dt \right\} \end{bmatrix}$$

so that

$$|N\phi_1|^2 \leqslant |(H^2+p)F - F^{(2)}|^2 + |(J^2+q)G - G^{(2)}|^2 + |rG|^2 + |rF|^2 + |2HF^{(1)} + H^{(1)}F|^2 + |2JG^{(1)} + J^{(1)}G|^2. \qquad \ldots(3.4)$$

From (3.2) and (3.4) it follows that ϕ_1 is in D if

(a) $\begin{pmatrix} F \\ G \end{pmatrix}$ is $L^2[0, \infty)$

(b) $r\begin{pmatrix} F \\ G \end{pmatrix}$ is $L^2[0,\infty)$

(c) $\begin{pmatrix} 2HF^{(1)} + H^{(1)}F \\ 2JG^{(1)} + J^{(1)}G \end{pmatrix}$ is $L^2[0,\infty)$

(d) $\begin{pmatrix} (H^2+p)F - F^{(2)} \\ (J^2+q)G - G^{(2)} \end{pmatrix}$ is $L^2[0, \infty)$,

We may note that the interval $[0, X]$ is of little importance in the conditions (a), (b), (c) and (d) above as we consider the behaviour as $x \to \infty$.

Let us assume that in $[X, \infty)$

$$F(x) = x^{-\alpha/4}(1 + S_1(x)),\ H(x) = a^{1/2} x^{\alpha/2} / (1+S_1(x))^2 \qquad \ldots(3.5)$$
$$G(x) = x^{-\beta/4}(1 + S_2(x)),\ J(x) = b^{1/2} x^{\beta/2} / (1+S_2(x))^2.$$

$$\left.\begin{aligned} &\text{Then } HF^2 = a^{1/2} \text{ and so } 2HF^{(1)} + H^{(1)}F = \frac{(HF^2)^{(1)}}{F} = 0 \\ &\text{and similarly} \quad 2JG^{(1)} + J^{(1)}G = 0. \end{aligned}\right\} \qquad \ldots(3.6)$$

$$\left.\begin{aligned}(H^2+p)\,F-F^{(2)} &= ax\,\frac{3\alpha}{4}\,S_1.\,\frac{4+6S_1+4S_1^2+S_1^3}{(1+S_1)^3}-\frac{\alpha\,(\alpha+4)}{16}\,x^{-\alpha/4-2}\,(1+S_1)\\ &\quad+\frac{\alpha}{2}\,x^{-\alpha/4-1}S_1^{(1)}+x^{-\alpha/4}\,S_1\,S_1^{(2)}+h_1\,(1+S_1),\\ (J^2+q)\;G-G^{(2)} &= bx\,\frac{3\beta}{4}\,S_2.\,\frac{4+6S_2+4S_2^2+S_2^3}{(1+S_2)^3}-\frac{\beta(\beta+4)}{16}\,x^{-\beta/4-2}\,(1+S_2)\\ &\quad+\frac{\beta}{2}\,x^{-\beta/4-1}\,S_2^{(1)}+x^{-\beta/4}\,S_2\,S_2^{(2)}+h_2\,(1+S_2).\end{aligned}\right\}\quad\ldots(3.7)$$

Hence by (3.5)

$$\begin{pmatrix} F \\ G \end{pmatrix} \text{ and } r\begin{pmatrix} F \\ G \end{pmatrix} \text{ are } L^2\,[X,\infty)$$

and by (3.6)

$$\begin{pmatrix} 2\,HF^{(1)}+H^{(1)}\,F \\ 2\,JG^{(1)}+J^{(1)}G \end{pmatrix} \text{ is } L^2\,[X,\infty).$$

By (3.7) and conditions (ii), (iii) and (iv)

$$\begin{pmatrix} (H^2+p)\,F-F^{(2)} \\ (J^2+q)\,G-G^{(2)} \end{pmatrix} \text{ is } L^2\,[X,\infty).$$

Thus with our choice of F, G, H, and J as in (3.5) ϕ_1 is in D and

$[\phi_1\,\phi_1]\,(x) = -\,2i\,(a^{1/2}+b^{1/2})$, which does not tend to zero as $x\to\infty$.

Hence N is not limit-2.

Now it remains to prove that if $r\,(x) = O\,(1)$ and at least one of $p\,(x)$ and $q\,(x)$ is non-negative, N is limit-3. If all the solutions of (1.2) are $L^2\,[0,\infty)$ for one value of λ (complex or real), it is so for any value of λ [Bhagat (1969b) and Naimark (1968, p. 93)]. We take $\lambda = 0$. $r(x) = O\,(1)$, without loss of generality we can take $r = 0$. Then following Titchmarsh (1962, § 5.4) we can prove that there is a solution $u\,(x)$ of

$$u^{(2)}\,(x) + p\,(x)\,u\,(x) = 0$$

such that $u\,(x)\to\infty$ as $x\to\infty$ if $p\,(x) \geqslant 0$, and consequently there is a solution of (1.2) which is not $L^2\,[0,\infty)$.

This completes the proof of the theorem.

§ 4. Let

$$S_1\,(x) = -\,k_1\,\mu^{-2}\,x^{\nu_1-2(\mu_1-1)}\sin\,(x_{\mu_1})$$

and

$$S_2\,(x) = -\,k_2\,\mu^{-2}\,x^{\nu_2-2\,(\mu_2-1)}\sin\,(x^{\mu_2})$$

where μ_1, μ_2, ν_1 and ν_2 are all positive and k_1 and k_2 are constants.

Then

$$S_1^{(2)}\,(x) = k_1\,x^{\nu_1}\sin\,(x^{\mu_1}) + T_1\,(x), \qquad\ldots(4.2)$$

and

$$S_2^{(2)}\,(x) = k_2\,x^{\nu_2}\sin\,(x_2) + T^{\mu}{}_2\,(x),$$

where $T_1(x) =, O(x^{\nu_1 - \mu_1})$ and $T_2(x) = O(x^{\nu_2 - \mu_2})$, ...(4.3)
as $x \to \infty$.
Condition (ii) of the theorem holds if

$$\nu_1 - 2(\mu_1 - 1) < 0, \quad ...(4.4)$$

and $\nu_2 - 2(\mu_2 - 1) < 0$, .. (4.5)
and conditions (iii) hold if

$$\frac{3\alpha}{4} + \nu_1 - 2(\mu_1 - 1) < -\frac{1}{2} \quad ...(4.6)$$

$$\frac{3\beta}{4} + \nu_2 - 2(\mu_2 - 1) < -\frac{1}{2} \quad ...(4.7)$$

$$-\frac{\alpha}{4} - 1 + \nu_1 - (\mu_1 - 1) < -\frac{1}{2} \quad ...(4.8)$$

$$-\frac{\beta}{4} - 1 + \nu_2 - (\mu_2 - 1) < -\frac{1}{2} \quad ...(4.9)$$

$$-\frac{\alpha}{4} + 2\nu_1 - 2(\mu_1 - 1) < -\frac{1}{2} \quad ...(4.10)$$

$$-\frac{\beta}{4} + 2\nu_2 - 2(\mu_2 - 1) < -\frac{1}{2}. \quad ...(4.11)$$

If we choose $h_1(x) = x^{-\alpha/4} T_1(x)$ and $h_2(x) = x^{-\beta/4} T_2(x)$,
then condition (iv) of the theorem holds if

$$-\frac{\alpha}{4} + \nu_1 - \mu_1 < -\frac{1}{2} \quad ...(4.12)$$

and

$$-\frac{\beta}{4} + \nu_2 - \mu_2 < -\frac{1}{2} \quad ...(4.13)$$

(4.12) and (4.13) are the same as (4.8) and (4.9) respectively and they hold if (4.10) and (4.11) hold. (4.4) and (4.5) hold if (4.6) and (4.7) respectively hold. Thus we have the following theorem.

Theorem—Let a, b, α and β be as in the theorem of § 3. Let $\mu_1, \mu_2 > 0$,

$$0 < \nu_1 < \min\left((\mu_1 + \frac{\alpha}{8} - \frac{5}{4}, 2\mu_1 - \frac{3\alpha}{4} - \frac{5}{2} \right),$$

...(4.14)

$$0 < \nu_2 < \min\left((\mu_2 + \frac{\beta}{8} - \frac{5}{4}, 2\mu_2 - \frac{3\beta}{4} - \frac{5}{2} \right)$$

and

$$0 > \gamma > \frac{2}{\alpha}.$$

Then if

$$p(x) = -ax^{\alpha} + k_1 x^{\nu_1} \sin(x^{\mu_1})$$
$$q(x) = -bx^{\beta} + k_2 x^{\nu_2} \sin(x^{\mu_2})$$

and

$r(x) = cx^{\gamma} \sin(x^{\delta})$, c being a constant and $\delta \in (-\infty, \infty)$, then N is not limit-2.

When $\nu_1 = \alpha$, and $\nu_2 = \beta$, conditions in (4.14) reduce to

$$\mu_1 > \frac{7}{8}\alpha + \frac{5}{4} \text{ and } \mu_2 > \frac{7}{8}\beta + \frac{5}{4}.$$

Since α, $\beta > 2$, the ranges for μ_1 and μ_2 lie in the interval $(3, \infty)$.

Corollary—Let a, b, α, β, ν_1, ν_2, μ_1 and μ_2 be as in the theorem and $\gamma < 0$. In addition, let $\nu_1 > \alpha$ and $k_1 > a$ or $\nu_2 > \beta$ and $k_2 > b$. Then

$$p(x) = -ax^{\alpha} + k_2 x^{\nu_1} \mid \sin(x^{\mu_1}) \mid$$
$$q(x) = -bx^{\beta} - k_2 x^{\nu_2} \mid \sin(x^{\mu_2}) \mid$$

and $$r(x) = cx^{\gamma} \sin(x^{\delta})$$

make N limit-3.

§ 5. In this section we are extending the result of Titchmarsh (1962, § 2.20) to the system of eqns. (1.2). As the method of proof is the same as that of Titchmarsh, the details are omitted.

Theorem—Let $p(x), q(x) \geqslant -P(x)$ where $P(x)$ is positive, continuous and non-decreasing, and the integral

$$\int_0^\infty [P(x)]^{-1/2}\, dx$$

is divergent and $r(x) = O(1)$. Then N is limit-2.

Let $\phi_j \equiv \phi_j(0/x, \lambda)$ $(j = 1,2)$ be the boundary condition vectors at $x = 0$ and $\theta_k \equiv \theta_k(0/x, \lambda) = \begin{pmatrix} x_k(0/x, \lambda) \\ y_k(0/x, \lambda) \end{pmatrix}$ be the solutions of (1.2) such that

$$[\phi_j\, \theta_k] = \delta_{jk}, \ (1 \leqslant j, k \leqslant 2)$$

and $$[\theta_1\, \theta_2] = 0.$$

Then the Wronskian of ϕ_1, ϕ_2, θ_1 and θ_2 is unity and so they form a fundamental set of solutions for the system (1.2) (see Bhagat 1969b, § 4). Then it is sufficient to show that one of ϕ_1 and θ_1, and one of ϕ_2 and θ_2 are not $L^2[0, \infty)$.

For any two real vectors $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, we have

$$(F_1 G_1 + F_2 G_2)^2 \leqslant (F_1^2 + F_2^2)(G_1^2 + G_2^2)$$

whence

$$\left| \int_0^\infty F^T G\, dx \right| \leqslant \int_0^\infty (F_1^2 + F_2^2)^{1/2} (G_1^2 + G_2^2)^{1/2}\, dx \leqslant \left\{ \int_0^\infty F^T F\, dx \int_0^\infty G^T G\, dx \right\}^{1/2}, \quad \ldots(5.1)$$

by Schwartz inequality.

First suppose that ϕ_1 and θ_1 are $L^2[0, \infty)$.

Since $[\phi_1\, \theta_1] = 1$, we have

$$\int_0^\infty [P(2x)]^{-\frac{1}{2}}\, dx = \int_0^\infty [P(2x)]^{-\frac{1}{2}} \left[u_1{}^{(1)}{}_{\bar{x}_1} - u_1{}_{\bar{x}_1}{}^{(1)} + v_1{}^{(1)}{}_{\bar{y}_1} - v_1{}_{\bar{y}_1}{}^{(1)} \right] dx$$

$$\leqslant \left\{ \mid \theta_1 \mid^2 dx \int_0^\infty \mid \phi_1^{(1)} \mid^2 [P(2x)]^{-1}\, dx \right\}^{\frac{1}{2}} + \left\{ \int_0^\infty \mid \phi_1 \mid^2 dx \int_0^\infty \mid \theta_1^{(1)} \mid^2 [P(2x)]^{-1}\, dx \right\}^{\frac{1}{2}}$$

by the inequality (5.1).

In order to obtain a contradiction it is sufficient to prove that the integrals

$$\int_0^\infty |\phi_1^{(1)}|^2 [P(2x)]^{-1} dx \quad \text{and} \quad \int_3^\infty | \theta_1^{(1)}|^2 [P(2x)]^{-1} dx$$

are convergent.

$$\text{Let} \quad C = \int_0^\infty |\phi_1|^2 dx, \; I = \int_0^X \left(1 - \frac{x}{X}\right)^2 |\phi_1^{(1)}|^2 dx, J = \int_0^X |\phi_1|^2 P(x) dx,$$

Then following Titchmarsh (1962, § 2.20) it can be shown that

$$(I^{\frac{1}{2}} - X^{-1} C^{\frac{1}{2}})^2 \leqslant K + C|\lambda| + J + RC + \frac{C}{X^2},$$

where $k = \phi_1^T(0, \lambda) \phi_1^{(1)}(0, \overline{\lambda})$ and $|r(x)| \leqslant R$.

Then again following the same method it can be proved that

$$\int_0^X |\phi_1^{(1)}|^2 [P(2x)]^{-1} dx \text{ is convergent, as } X \to \infty,$$

and similarly $\int_0^X |\theta_1^{(1)}|^2 [P(2x)]^{-1} dx$ is convergent, as $X \to \infty$.

Hence ϕ_1 and θ_1 are not both $L^2[0, \infty)$; that is at least one of them is not $L^2[0, \infty)$. Similarly, at least one of ϕ_2 and θ_2 is not $L_2^2[0, \infty)$. But by § 1, equation (1. 2) has at least two linearly independent solutions belonging to $L^2[0, \infty)$. So one of ϕ_1, θ_1 and one of ϕ_2, θ_2 are $L^2[0, \infty)$.

Thus two linearly independent solutions of (1.2) are not $L^2[0, \infty)$, and so N is limit-2.

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