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ON DOUBLE INTEGRAL TRANSFORMS

MRIDULA GARG

Department of Mathematics, M.R. Engineering College, Jaipur 302004

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In this paper we derive two new and interesting theorems interconnecting the images of related functions in two general integral transforms. Certain special cases of these theorems, which are new and of interest in themselves, have also been mentioned. Exact references of ten known theorems, which are obtainable as particular cases of our theorems, have also been included. Finally, an infinite integral has been evaluated as an application of one of the theorems: this integral is believed to be new.

1. INTRODUCTION

A large number of theorems have been obtained involving various integral transforms. Recently, Gupta (1980) gave four theorems which are quite general in character. Our main Theorems I, II, II(a) and II(b) not only further generalize two of the theorems of Gupta (1980), but also include, as their particular cases, a number of known or new theorems. Finally, we evaluate a new integral by the application of Theorem II (b), which is a corollary of Theorem II.

The following definitions and results will be used in the sequel.

(a) (i) *Laplace transform:*

$$L\{f(x); p\} = \int_0^\infty e^{-px} f(x)\, dx\,. \qquad \ldots(1.1)$$

(ii) *Meijer Bessel-function transform:*

$$K\{f(x); \nu; p\} = \int_0^\infty (px)^{1/2} K_\nu(px) f(x)\, dx\,. \qquad \ldots(1.2)$$

(iii) *Varma transform:*

$$W\{f(x)\,; k, m; p\} = \int_0^\infty (px)^{m-1/2} e^{-px/2} W_{k,m}(px) f(x) dx\,. \qquad \ldots(1.3)$$

(iv) *Stieltjes transform:*

$$S\{f(x)\,; p\} = \int_0^\infty (x+p)^{-1} f(x)\, dx\,. \qquad \ldots(1.4)$$

(b) The generalized Parseval-Goldstein formula:

$$\int_0^\infty \int_0^\infty f_1(x,y)\, \phi_2(x,y)\, dx\, dy = \int_0^\infty \int_0^\infty f_2(x,y)\, \phi_1(x,y)\, dx\, dy \qquad \ldots(1.5)$$

where

$\phi_1(p,q) = T\{ f_1(x,y) ; p, q\}$

and $\phi_2(p,q) = T\{ f_2 (x,y) ; p, q\}$.

(c) A result given by Srivastava [1974, p. 827, eqn. (2.4)]

$$p^{-\sigma-1}e^{\alpha p}W_{k,m}(2\alpha p)G^{M,N}_{P,Q}\left[\frac{a}{p}\,\middle|\,\begin{matrix}(a_j)_{1,P}\\(b_j)_{1,Q}\end{matrix}\right]$$

$$=L\Big\{(2a)^{\sigma}x^{-k+\sigma}(1+x)^{k+\sigma}\,G^{0,2:1,0;M,N}_{2,1:0,1;P,Q+2}\left[\begin{matrix}x\\2a\alpha x(1+x)\end{matrix}\,\middle|\right.$$

$$\pm\, m-\sigma+1/2:—;\ (a_j)_{1,P}$$

$$\left.k-\sigma:0;(b_j)_{1,Q},\ \pm m-\sigma+1/2\right];p\Big\} \qquad \ldots(1.6)$$

in terms of the G-function of two variables, it being provided that $\alpha>0$, $\text{Re}(p) > 0$, $\text{Re}(b_j+\sigma+1-k)>0, j=1,\ldots\ldots,M$

$2(M+N)>P+Q+1$, $|\arg a| <(M+N-P/2-Q/2-1/2)\pi$.

(d) A special case of the result given by Gupta and Mittal [1972, p. 121, eqn. (2.3)]

$$\int_0^\infty x^{\lambda-1}e^{-ax/2}W_{k,m}(ax)\,H^{M,N}_{P,Q}\left[\beta x^h\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right]$$

$$\times\, H^{M',N'}_{P',Q'}\left[\delta x^{\sigma}\,\middle|\,\begin{matrix}(c_j,\gamma_j)_{1,P'}\\(d_j,\delta_j)_{1,Q'}\end{matrix}\right]dx$$

$$=a^{-\lambda}H^{0,2:M,N;M',N'}_{2,1:P,Q;P',Q'}\left[\begin{matrix}\beta/a^h\\\delta/a^{\sigma}\end{matrix}\,\middle|\,\begin{matrix}(1/2\pm m-\lambda;\ h,\sigma):(a_j,\alpha_j)_{1,P};(c_j,\gamma_j)_{1,P'}\\(k-\lambda;\ h,\sigma)\qquad:(b_j,\beta_j)_{1,Q};(d_j,\delta_j)_{1,Q'}\end{matrix}\right] \qquad \ldots(1.7)$$

provided that

$$\text{Re}\left\{\lambda\pm m+\tfrac{1}{2}+h\left(\frac{b_j}{\beta_j}\right)+\sigma\left(\frac{d_{j'}}{\delta_{j'}}\right)\right\} > 0,\ j=1,\ldots,M;\ j'=1,\ldots,N';$$

h and σ are positive quantities,

$$\lambda_1=\sum_{j=1}^{M}\beta_j-\sum_{j=M+1}^{Q}\beta_j+\sum_{j=1}^{N}\alpha_j-\sum_{j=N+1}^{P}\alpha_j > 0$$

$$\lambda_2=\sum_{j=1}^{M'}\delta_j-\sum_{j-M'+1}^{Q'}\delta_j+\sum_{j=1}^{N'}\gamma_j-\sum_{j=N'+1}^{P'}\gamma_j > 0,$$

$$|\arg\beta| < \tfrac{1}{2}\lambda_1\pi,\quad |\arg\delta| < \tfrac{1}{2}\lambda_2\pi.$$

(e) A result due to Gupta [1965, p. 100, eqn. (8)]

$$L\left\{x^l H^{m,n}_{p,q}\left[zx^{\sigma}\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right];\ p'\right\}$$

$$=p^{-l-1}H^{m,n+1}_{p+1,q}\left[zp'^{-\sigma}\,\middle|\,\begin{matrix}(-l,\sigma),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]$$

where $\sigma > 0$, $\text{Re}(p) > 0$, $\text{Re}\left[1+1+\sigma\left(\frac{b_j}{\beta_j}\right)\right] > 0$, $\quad j = l,\ldots,m$,

$$\lambda' = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j > 0$$

$$|\arg z| < \tfrac{1}{2}\lambda'\pi . \qquad \ldots(1.8)$$

It may be of interest to remark in passing that, for the G- and H- functions of two variables occurring in this paper, we freely employ the contracted notations [due essentially to Srivastava and Joshi (1969)] which were formally introduced by Srivastava and Panda [1976, p. 266, eqn. (1.5) ; p. 267, eqn. (1.11)].

2. Theorems

Theorem I—If

$$h_1(p, q) = T_1\{g(x, y) f(x, y) ; p, q\}$$
$$= \int_0^\infty \int_0^\infty k_1(px, qy)\, g(x,y)\, f(x, y)\, dx\, dy \qquad \ldots(2.1)$$

and

$$h_2(p, q) = T_2\{f(x^{-\sigma}, y^{-\mu}) ; p, q\}$$
$$= \int_0^\infty \int_0^\infty k_2(px, qy)\, f(x^{-\sigma}, y^{-\mu})\, dx\, dy, \qquad \ldots(2.2)$$

then

$$h_1(p, q) = \sigma\mu \int_0^\infty \int_0^\infty h_2(x, y)\, \phi(x, y, p, q)\, dx\, dy \qquad \ldots(2.3)$$

where

$$p^{-\sigma-1} q^{-\mu-1} g(p^{-\sigma}, q^{-\mu})\, k_1(\alpha p^{-\sigma}, \beta q^{-\mu})$$
$$= T_2\{\phi(x, y, \alpha, \beta) ; p, q\} . \qquad \ldots(2.4)$$

σ, μ are non-zero real numbers of the same sign, α, β are independent of p and q, and the integrals involved in (2.1) to (2.4) are assumed to be absolutely convergent.

Proof : Applying the generalized Parseval-Goldstein formula given by (1.5) to the pairs (2.2) and (2.4), we obtain

$$\int_0^\infty \int_0^\infty x^{-\sigma-1} y^{-\mu-1} g(x^{-\sigma}, y^{-\mu})\, k_1(\alpha x^{-\sigma}, \beta y^{-\mu})\, f(x^{-\sigma}, y^{-\mu})\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty h_2(x,y)\, \phi(x,y,\alpha,\beta)\, dx\, dy .$$

Now changing the variables slightly on the left-hand side of the above equation and interpreting the result thus obtained in terms of (2.1), we easily arrive at the required result (2.3) after replacing α by p and β by q.

If, in Theorem I, we take the transforms T_1 and T_2 to be analogous transforms involving one variable, we shall arrive at the following theorem for one variable.

Theorem II – If

$$h_1(p) = T_1\{g(x) f(x) ; p\} = \int_0^\infty k_1(px)\, g(x)\, f(x)\, dx \qquad \ldots(2.5)$$

and $h_2(p) = T_2\{f(x^{-\sigma}); p\} = \int_0^\infty k_2(px) f(x^{-\sigma})\, dx$ ...(2.6)

then $h_1(p) = |\sigma| \int_0^\infty h_2(x)\, \phi(x,p)\, dx$ ...(2.7)

where $p^{-\sigma-1} g(p^{-\sigma})\, k_1(\alpha p^{-\sigma}) = T_2\{\phi(x,\alpha)\,;\, p\}$ ...(2.8)
σ is a non-zero real number, α is independent of p, and the integrals involved in (2.5) to (2.8) are assumed to be absolutely convergent.

If, in Theorem II, we take T_2 as the Laplace transform defined by (1.1), we get the following interesting form of the theorem after a little simplification:

Theorem II (a)—If

$h_1(p) = T_1\{f(x)\, g(x)\,;\, p\}$ ...(2.9)

and $h_2(p) = L\{f(x^{-\sigma})\,;\, p\}$, ...(2.10)

then $h_1(p) = |\sigma| \int_0^\infty h_2(x+p)\, \phi(x,p)\, dx$ ...(2.11)

where $e^{\alpha p}\, p^{-\sigma-1} g(p^{-\sigma})\, k_1(\alpha p^{-\sigma}) = L\{\phi(x,\alpha)\,;\, p\}$ ...(2.12)
Re $(p)>0$, σ is a non-zero real number, α is independent of p, and the various integrals in (2.9) to (2.12) are assumed to be absolutely convergent.

If, in Theorem II(a), we take T_1 to be the Varma transform defined by (1.3), $\sigma=-1$ and

$$g(x) = x^{-\rho-m-1/2}\, G^{M,N}_{P,Q}\left[\frac{a}{x}\,\middle|\, \begin{matrix}(a_j)_{1,P}\\ (b_j)_{1,Q}\end{matrix}\right] \qquad ...(2.13)$$

we get, on using the result (1.6) and after a little simplification, a new and interesting theorem given below.

Theorem II (b)—If

$$h_1(p) = V\left\{x^{-\rho-m-1/2}\, G^{M,N}_{P,Q}\left[\frac{a}{x}\,\middle|\, \begin{matrix}(a_j)_{1,P}\\ (b_j)_{1,Q}\end{matrix}\right] f(x);\, k,\, m; p\right\} \qquad ...(2.14)$$

and $h_2(p)=L\{f(x)\,;\,p\}$, ...(2.15)

$$\text{then } h_1(p) = (2p)^{\rho+m-1/2} \int_0^\infty x^{-k+\rho}(1+x)^{k+\rho}\; G^{0,2:1,0;M,N}_{:,1,0:1,P,Q+2}\left[\begin{matrix}x\\ 2apx(1+x)\end{matrix}\,\middle|\, \begin{matrix}\pm m-\rho+\tfrac{1}{2}:-; & (a_j)_{1,P}\\ k-\rho;0; & (b_j)_{1,Q},\pm m-\rho+\tfrac{1}{2}\end{matrix}\right] h_2(p+x)\, dx \qquad ...(2.16)$$

where

Re$(p)>0$, Re $(b_j+\rho+1-k) > 0$, $j = 1,\ldots, M$ $2(M+N)>P+Q+1$,
$|\arg a| <(M+N-P/2-Q/2-1/2)\,\pi$, and the integral in (2.16) is assumed to be absolutely convergent.

On making suitable choices of transforms T_1, T_2, σ and $g(x)$, Theorems I, II and II(a) reduce to a number of theorems established earlier by various authors. We list some such special cases of our theorems in the next section.

3. Special Cases

(i) If in Theorem II, we take $\sigma=-\frac{1}{2}$, the transform T_1 as defined by (1.2), T_2 as the transforms defined by (1.1) and (1.4), respectively, and the function $g(x)$ as $x^{1/2}I_\nu(\delta x)$ and $x^{\nu-\mu+n+1/2}I_\nu(ax)$ in succession, we easily get after a little simplification the two theorems obtained by Maloo [1966 b, eqn. (5); 1967 c, eqn. (5)].

(ii) If in Theorem II(a), we take $\sigma=-1$, the transform T_1 to be the Varma transform defined by (1.3) and $g(x)$ to be $x^{-m}K_{2m}(ax^{1/2})$ and $x^{-m}I_{2m}(ax^{1/2})$ in succession, we easily get the two theorems obtained by Saxena [1964, Theorems I and II].

(iii) If in Theorem II(a), we take $\sigma=-1$, the transform T_1 to be the Laplace transform defined by (1.1) and $g(x)$ to be as $x^{-\lambda-2\alpha+\beta}I_\beta(a/x)$ and $x^{-\sigma}e^{ax/2}W_{k,\mu}(ax)$ in succession, we get the other two theorems obtained by Maloo [1966 a, Theorems 1 and 2].

(iv) In the case when $g(x)$ is taken to be $x^{(\rho/\sigma)-1}$, our Theorems I and II, yield the theorems obtained earlier by Gupta [1980, Theorems II and IV].

(v) If in Theorem II (a) we take $\sigma=-1$, T_1 as the transforms defined by (1.2) and (1.3), respectively, and the function $g(x)$ as $x^{-1/2}e^{\beta x}K_\nu(\beta x)$ and $x^{-m-1/2}e^{\beta x/2}W_{\lambda,m}(\beta x)$ in succession, we get two other theorems of Maloo [1967b, eqn. (3); 1967 a, eqn. (3)].

4. An Application of Theorem II (b)

In this section we shall evaluate an infinite integral with the application of Theorem II (b). We take

$$f(x)=x^n H_{P',Q'}^{M',N'}\left[bx^\sigma \,\middle|\, \begin{matrix}(c_j,\gamma_j)_{1,P'}\\(d_j,\delta_j)_{1,Q'}\end{matrix}\right] \qquad \ldots(4.1)$$

where the H-function on the right-hand side of the above equation is the well-known (Fox's) H-function. [see, for example, Gupta and Jain (1966)].

Then, on applying the result (1.8), we get

$$h_2(p)=p^{-n-1}H_{P'+1,Q'}^{M',N'+1}\left[bp^{-\sigma}\,\middle|\, \begin{matrix}(-n,\sigma),(c_j,\gamma_j)_{1,P'}\\(d_j,\delta_j)_{1,Q'}\end{matrix}\right] \qquad \ldots(4.2)$$

which is valid under the conditions stated with (1.8). Further, on using (2.14) and the result (1.7), we get

$$h_1(p)=p^{m+\rho-n-(1/2)}\,H^{0,2;N,M;\,M',N'}_{2,1:Q,P;\,P',Q'}\left[\begin{matrix}1/ap\\b/p^\sigma\end{matrix}\,\middle|\, \begin{matrix}(\frac{1}{2}\pm m-n+\rho;\,1,\sigma):(1-b_j,1)_{1,Q};(c_j,\gamma_j)_{1,P'}\\(k-n+\rho;\,1,\sigma):(1-a_j,1)_{1,P};(d_j,\delta_j)_{1,Q'}\end{matrix}\right] \qquad \ldots(4.3)$$

which is valid under the conditions easily obtainable from those stated with (1.7).

Now substituting the values of $h_1(p)$ and $h_2(p)$ from equations (4.3) and (4.2) in equation (2.16) of Theorem II (b), we get the following new integral.

$$\int_0^\infty x^{-k+\rho}(1+x)^{k+\rho}(p+x)^{-n-1}\,H_{P'+1,Q'}^{M',N'+1}\left[\frac{b}{(p+x)^\sigma}\,\middle|\right.$$

(equation continued on p. 445)

$$\left.\begin{matrix}(-n,\sigma), (c_j,\gamma_j)_{1,P'} \\ (d_j, \delta_j)_{1,Q'}\end{matrix}\right] G^{0,2:\ ,0;M,N}_{2,1:0,1;P,Q+2}\left[\begin{matrix}x \\ 2\,apx(1+x)\end{matrix}\right|$$

$$\left.\begin{matrix}\pm m-\rho+\tfrac{1}{2}:\ -;\ (a_j)_{1,P} \\ k-\rho:\ 0;\ (b_j)_{1,Q},\pm m\ \rho+\tfrac{1}{2}\end{matrix}\right]dx = 2^{1/2-\rho-m}\ p^{-n}\ H^{0,2:N,M;\ M',N'}_{2,1:Q,P;P',Q'}\left[\begin{matrix}1/ap \\ b/p^{\sigma}\end{matrix}\right|$$

$$\left.\begin{matrix}(\tfrac{1}{2}\pm m-n+\rho;\ 1,\sigma):\ (1-b_j,1)_{1,Q};\ (c_j,\gamma_j)_{1,P'} \\ (k-n+\rho\ ;\ 1,\sigma):\ (1-a_j,\ 1)_{1,P};\ (d_j,\delta_j)_{1,Q'}\end{matrix}\right] \qquad \ldots(4.4)$$

where $\sigma>0$, $\mathrm{Re}\,(k-\rho-1) < \min\{\mathrm{Re}(b_j)\}, j=1,\ldots,M$

$$\mathrm{Re}\left(\frac{n}{2}-\rho\right)>\max\left\{\mathrm{Re}\left[a_j-1-\frac{\sigma}{2}\left(\frac{d_{j'}}{\delta_{j'}}\right)\right]\right\},\ j=1,\ldots,N,\ j'=1,\ldots,M'$$

$$\lambda''=\sum_{j=1}^{M'}\delta_j-\sum_{j=M'+1}^{Q'}\delta_j+\sigma+\sum_{j=1}^{N'}\gamma_j-\sum_{j=N'+1}^{P'}\gamma_j>0,$$

$$|\arg b| < \tfrac{1}{2}\lambda''\pi,$$

$$2(M+N)>P+Q+1,\quad |\arg a| < \tfrac{1}{2}(M+N-P/2-Q/2-\tfrac{1}{2})\pi.$$

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