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ON SEQUENCE-TO-SEQUENCE TRANSFORMATIONS

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In this paper we extend the sequence-to-sequence Hausdorff transformations to functional sequences and introduce a new integral transform which arises during the course of our study.

1. INTRODUCTION

The sequence-to-sequence transformations were introduced by Hausdorff as the set of methods

$$t_n = \sum_{m=1}^{\infty} C_{mn} S_m \qquad (n = 1, 2, \ldots), \qquad \ldots(1.1)$$

where $\{S_m\}_{m=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are two sequences and a_{mn} $(m, n = 1, 2, \ldots)$ is a set of real numbers which can be arranged in the form of a matrix $\| a_{mn} \|$. The sequence $\{S_n\}_{n=1}^{\infty}$ is said to be summed to S by the matrix $\| a_{mn} \|$, if the sum on the right-hand side of (1.1) exists and if $\lim\limits_{n\to\infty} t_n = S$.

In this paper the transformations (1.1) are extended for functional sequences. Conditions for regularity of the methods are obtained and a new integral transform, which arises as a natural consequence, is also introduced.

2. THE SEQUENCE-TO-SEQUENCE TRANSFORMATIONS

Suppose there are two functional sequences $\{ f_n(x) \}_{n=1}^{\infty}$ and $\{ t_m(x) \}_{m=1}^{\infty}$. Let there be a set of real-valued functions $\mu_{mn}(x)$; $(m, n = 1, 2, \ldots)$ which can be arranged in the form of a matrix

$$\| \mu_{mn}(x) \| = \left\| \begin{array}{cccc} \mu_{11}(x) & \mu_{12}(x) & \ldots & \ldots \\ \mu_{21}(x) & \mu_{22}(x) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array} \right\| .$$

Then the transformations given by the set of methods

$$t_m(x) = \sum_{n=1}^{\infty} \mu_{mn}(x) f_n(x) \qquad (m = 1, 2, \ldots; x > 0) \qquad \ldots(2.1)$$

are the extended forms of the well known Hausdorff transformations (1.1).

The sequence $\{f_n(x)\}_{n=1}^{\infty}$ is said to be summed to $f(x)$ by the matrix $\| \mu_{mn}(x) \|$ if the sum on the right-hand side of (2.1) exists and if $\lim_{m\to\infty} t_m(x) = f(x)$.

Theorem 2.1—The transformations (2.1) are regular if and only if

$$\text{(A)} \quad \lim_{m\to\infty} \sum_{n=1}^{\infty} \mu_{mn}(x) = 1 \qquad (x > 0),$$

$$\text{(B)} \quad \lim_{m\to\infty} \mu_{mn}(x) = 0 \qquad (n = 1, 2, \ldots ; x > 0),$$

$$\text{(C)} \quad \sum_{n=1}^{\infty} | \mu_{mn}(x) | = M \qquad (m = 1, 2, \ldots; x > 0);$$

where M is any constant.

PROOF: First we prove the necessity. Let the transformations (2.1) be regular (that is, the sequence $\{t_m(x)\}_{m=1}^{\infty}$ converges to $f(x)$, whenever $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$).

Now let $\quad f_n(x) = f(x) + \delta_n(x), \qquad \ldots(2.2)$

where $\quad \delta_n(x) \to 0 \quad$ as $f_n(x) \to f(x)$

and let $\quad t_m(x) = f(x) + \epsilon_m(x), \qquad \ldots(2.3)$

where $\quad \epsilon_m(x) \to 0 \quad$ as $t_m(x) \to f(x)$.

Then by the regularity of (2.1) it is clear that the sequence $\{\epsilon_m(x)\}_{m=1}^{\infty}$ converges to zero, whenever the sequence $\{\delta_n(x)\}_{n=1}^{\infty}$ converges to zero.

Also with the help of (2.1), (2.2) and (2.3) we have

$$f(x) + \epsilon_m(x) = \sum_{n=1}^{\infty} \mu_{mn}(x) \, [f(x) + \delta_n(x)]$$

or $$f(x) \left[\sum_{n=1}^{\infty} \mu_{mn}(x) - 1 \right] + \sum_{n=1}^{\infty} \mu_{mn}(x) \, \delta_n(x) = \epsilon_m(x).$$

Hence $$\lim_{m\to\infty} f(x) \left[\sum_{n=1}^{\infty} \mu_{mn}(x) - 1 \right] + \lim_{m\to\infty} \sum_{n=1}^{\infty} \mu_{mn}(x) \, \delta_n(x) = 0, \qquad \ldots(2.4)$$

whenever $\lim_{n\to\infty} \delta_n(x) = 0$. Since (2.4) is true for all values of x in $(0, \infty)$ and for every $m, n = 1, 2, \ldots$; we have $\qquad \ldots(2.5)$

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \mu_{mn}(x) = 1 \qquad (x > 0) \qquad \ldots(2.6)$$

and $\lim_{m\to\infty} \mu_{mn}(x) \, \delta_n(x) = 0$ for every $n \geqslant 1$ and $x > 0$. $\qquad \ldots(2.7)$

Equations (2.5) and (2.7), together, imply that

$$\lim_{m\to\infty} \mu_{mn}(x) = 0 \quad \text{for every } n \geqslant 1 \text{ and } x > 0. \qquad \text{...(2.8)}$$

Now it follows by (2.7) that the sequence $\left\{ \sum_{n=1}^{\infty} \mu_{mn}(x)\, \delta_n(x) \right\}_{m=1}^{\infty}$ converges to zero; hence it must be bounded and there exists a fixed positive number K, such that

$$\left| \sum_{n=1}^{\infty} \mu_{mn}(x)\, \delta_n(x) \right| < K \text{ for every } m \geqq 1 \text{ and } x > 0. \qquad \text{...(2.9)}$$

Also it follows by (2.5) that

$$\left| \sum_{n=1}^{\infty} \mu_{mn}(x)\, \delta_n(x) \right| \leqq \sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right| \left| \delta_n(x) \right| < K' \sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right|. \qquad \text{...(2.10)}$$

Hence by (2.9) and (2.10) we have

$$\sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right| = M \qquad (m = 1, 2, \ldots;\ x > 0) \qquad \text{...(2.11)}$$

where M is any constant. Equations (2.6), (2.8) and (2.11) prove the necessity.

Conversely, let the conditions A, B and C hold.

Then $t_m(x) - f(x) = \sum_{n=1}^{\infty} \mu_{mn}(x)\,[f_n(x) - f(x)] + f(x) \left[\sum_{n=1}^{\infty} \mu_{mn}(x) - 1 \right].$

Hence
$$\overline{\lim_{m\to\infty}} \left| t_m(x) - f(x) \right| \leqq \overline{\lim_{m\to\infty}} \sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right| \left| f_n(x) - f(x) \right|$$
$$+ \left| f(x) \right| \overline{\lim_{m\to\infty}} \left| \sum_{n=1}^{\infty} \mu_{mn}(x) - 1 \right|.$$

With the help of condition A we obtain

$$\overline{\lim_{m\to\infty}} \left| t_m(x) - f(x) \right| \leqq \overline{\lim_{m\to\infty}} \sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right| \left| f_n(x) - f(x) \right|$$

$$= \overline{\lim_{m\to\infty}} \sum_{n=1}^{N} \left| \mu_{mn}(x) \right| \left| f_n(x) - f(x) \right| + \overline{\lim_{m\to\infty}} \sum_{n=N+1}^{\infty} \left| \mu_{mn}(x) \right| \left| f_n(x) - f(x) \right|,$$

where N is an arbitrary positive integer.
Now by conditions B and C

$$\overline{\lim_{m\to\infty}} \left| t_m(x) - f(x) \right| \leqq \underset{N+1 \leqq n < \infty}{u.\,b.} \left| f_n(x) - f(x) \right| \overline{\lim_{m\to\infty}} \sum_{n=N+1}^{\infty} \left| \mu_{mn}(x) \right|$$

$$\leqq M \underset{N+1 \leqq n < \infty}{u.\,b.} \left| f_n(x) - f(x) \right|.$$

Letting N to become infinite we get

$$\overline{\lim_{m\to\infty}} \left| t_m(x) - f(x) \right| \leqslant M \overline{\lim_{n\to\infty}} \left| f_n(x) - f(x) \right| = 0.$$

Hence $\lim_{m\to\infty} t_m(x) = f(x)$, whenever $\lim_{n\to\infty} f_n(x) = f(x)$.

This proves the sufficiency and completes our proof.

Theorem 2.2—The transformations (2.1) are regular, if and only if

$$\mu_{mn}(x) = \binom{m+n-1}{n-1} x^{n-1} \mathscr{B}_{mn}(x) \qquad \ldots(2.12)$$

$$\text{where} \quad \mathscr{B}_{mn}(x) = \int_0^\infty \frac{t^{n-1}}{(1+xt)^{m+n}} \, d\alpha(t), \qquad \ldots(2.13)$$

$\alpha(t) \in BV[0, \infty)$, $\alpha(0) = 0$ and $\alpha(\infty-) = 1$.

PROOF : To prove the theorem we have only to put the value of $\mu_{mn}(x)$ given by (2.12) in Theorem 2.1.

$$\text{Now} \sum_{n=1}^{\infty} \mu_{mn}(x) = \sum_{n=1}^{\infty} \binom{m+n-1}{n-1} x^{n-1} \mathscr{B}_{mn}(x)$$

$$= \int_0^\infty \frac{1}{(1+xt)^{m+1}} \sum_{n=1}^{\infty} \binom{m+n-1}{n-1} \left(\frac{xt}{1+xt}\right)^{n-1} d\alpha(t) = \int_0^\infty d\alpha(t).$$

$$\text{Hence} \quad \lim_{m\to\infty} \sum_{n=1}^{\infty} \mu_{mn}(x) = 1 \text{ and } \sum_{n=1}^{\infty} \left| \mu_{mn}(x) \right| \leqq \int_0^\infty \left| d\alpha(t) \right| < \infty.$$

Also, since $\alpha(t)$ is of bounded variation in $0 \leqq t < \infty$

$$\text{and} \binom{m+n-1}{n-1} (xt)^{n-1} \frac{1}{(1+xt)^{m+n}} \sim \frac{(xt)^{n-1}}{(n-1)!} \frac{m^{n-1}}{(1+xt)^{m+n}} = O(1) \qquad (m\to\infty);$$

We have $\lim_{m\to\infty} \mu_{mn}(x) = 0$. Thus conditions A, B, C of Theorem 2.1 are established and the proof is complete.

Remark : We may refer to equation (2.13) as the (m, n)th Beta transform or, from another point of view, as the (m, n)th Beta integral equation. Properties of this transform will be discussed later.

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