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ON MULTIPLIERS FOR THE ABSOLUTE NÖRLUND SUMMABILITY*

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Bhatt (1968) proved that Fourier series $\Sigma A_n \in |N, p_n|$ for $\varphi_1(t) \in BV(0,\pi)$ and $\{p_n\}$ obeys some restrictions. Under the same conditions we prove $\Sigma A_n \epsilon_n \in |N, p_n|$ for $\{\epsilon_n\} \in BV(0, \infty)$.

§1. If Σa_n be an infinite series with partial sums s_n. Then

$$t_n = (1/P_n) \sum_{k=0}^{n} p_{n-k} s_k \qquad \ldots(1.1)$$

defines Nörlund means where $\{p_n\}$ is a sequence of constants, real or complex and

$$P_n = p_0 + p_1 + \ldots + p_n \neq 0, \; P_{-1} = p_{-1} = 0 \qquad \ldots(1.2)$$

$$\text{If } \{t_n\}_0^\infty \in BV \text{ that is } \sum_{n=0}^{\infty} | t_n - t_{n-1} | < \infty \qquad \ldots(1.3)$$

we say that $a_n \in | N, p_n |$.

We assume that the sequence $\{p_n\}$ is non-negative and non-decreasing. Again let $f(t) \in L(o,\pi)$ and be 2π-periodic. Fourier series corresponding to $f(t)$ is

$$f(t) \sim 1/2\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \ldots(1.4)$$

We assume without loss of generality that the constant term is zero so that

$$\int_{\pi}^{\pi} f(t)dt = 0 \qquad \ldots(1.5)$$

We write $\varphi(t) = 1/2\{f(x+t) + f(x-t)\}$

$$\Phi_0(t) = \varphi(t)$$

$$\Phi_\alpha(t) = 1/(\Gamma\alpha) \int_0^t (t-u)^{\alpha-1} \varphi(u)\, du, \; \alpha > 0$$

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t), \; \alpha \geqslant 0$$

$$D_n(t) = 1/2 + \cos t + \cos 2t + \ldots + \cos nt = \frac{\sin (n+1/2)t}{2 \sin t/2}$$

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and $[x]$ denotes the integral part of x. C is written for an absolute constant but not the same at each occurrence.

§2. Generalising the theorem of Bosanquet (1936), Bhatt (1968) proved the following.

Theorem A (Bhatt 1968)—Let $\{p_n\}$ be a non-negative, non-decreasing sequence of real numbers such that

(i) $\{p_{n-1}-p_n\}$ is ultimately monotonic,

(ii) $\left\{\frac{(n+1)p_n}{P_n}\right\}\in BV$,

and (iii) $P_k/k \sum_{n=k}^{\infty} 1/P_n \leqslant C$

for $k=1, 2, 3,\ldots$ where C is a fixed constant. If $\varphi_1(t)\in BV(0,\pi)$, then (1.4) is summable $|N,p_n|$.

Remark 1: The necessary and sufficient conditions for regularity of (N,p_n)—method, are

$$\sum_{\nu=0}^{n} |p_\nu| = O(|P_n|) \qquad \ldots(1.6)$$

and $\quad p_n=o(|P_n|) \qquad \ldots(1.7)$

(1.6) is satisfied if $\{p_n\}$ is non-negative non-decreasing but (1.7) is not necessarily for example $p_n=2^n$. Of course (1.7) is satisfied by hypothesis (ii). Hence in Theorem A and later on in our case regularity of the method is maintained.

Following question is natural:

Let Σa_n is an infinite series absolutely summable by some summability methods. Is the series $a_n\epsilon_n$ also absolutely summable by the same method, when $\{\epsilon_n\}$ is a sequence of bounded variation? For a particular case $\epsilon_n\equiv 1$ the answer is trivially affirmative.

Suggested by this question it was observed that the sequences $\{\epsilon_n\}$ work as factor sequences for the result of Theorem A under the conditions stated therein. We call the elements of the set $\{\epsilon_n\}$ "multipliers". Following generalisation of Theorem A holds.

Theorem 1—We have

$$\varphi_1(t) \in BV(0,\pi) \Rightarrow \sum^{\infty}\epsilon_n A_n(t)\in |N,p_n| \qquad \ldots(2.2)$$

provided that $\{p_n\}$ is non-negative non-decreasing and satisfies the hypotheses (i), (ii), and (iii) of Theorem A and $\{\epsilon_n\}\in BV(0,\infty)$.

Following corollaries are noteworthy.

Corollary 1—Let $\{p_n\}$ satisfies all conditions of Theorem 1. Then

$$\varphi_1(t)\in BV(0,\pi) \Rightarrow \sum_{n=2}^{\infty} \frac{A_n(t)}{(\log n)^\delta}\in |N,p_n|, \ \delta\geqslant 0 \qquad \ldots(2.3)$$

Corollary 2—We have

$$\varphi_1(t)\in BV(0,\pi)\Rightarrow\sum_{n=2}^{\infty}\frac{A_n(t)}{(\log n)^{\delta}}\in|C,\alpha|,\alpha>1,\ \delta\geqslant 0. \qquad ...(2.4)$$

N.B. Corollary 2 is a particular case of Corollary 1, for $p_n=\binom{n+\alpha}{n}$, $\alpha>1$ which satisfy all conditions of Theorem A.

To complete the proof of Theorem 1 following lemmas are required.

Lemma 1 (Bhatt and Kishore 1967)—Let $\{p_n\}$ be any sequence of non-negative real members such that

$$\text{(i)}\quad\left\{\frac{(n+1)p_n}{P_n}\right\}\in BV \qquad ...(2.5)$$

$$\text{and (ii)}\quad P_k\sum_{n=k}^{\infty}\frac{1}{(n+1)P_n}\leqslant C. \qquad ...(2.6)$$

If $\sum_{k=0}^{n}a_k$ is bounded then a necessary and sufficient condition for $\sum_{k=0}^{\infty}a_k$ to be summable $|N,p_n|$ is that

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)P_n}\left|\sum_{k=1}^{n}p_{n-k}ka_k\right|<\infty. \qquad ...(2.7)$$

Lemma 2—If $\{q_n\}$ is non-negative and non-increasing then for $0\leqslant a\leqslant b\leqslant\infty$, $0\leqslant t\leqslant\pi$ and any n

$$\left|\sum_{k=a}^{b}q_k\epsilon_k e^{i(n-k)t}\right|\leqslant CQ_\tau; \qquad ...(2.8)$$

$$Q_m=q_0+q_1+\dots+q_m,$$

where $\tau=[1/t]$ and C is an absolute constant, $\{\epsilon_k\}\in BV(0,\infty)$.

Proof: We have $\epsilon_k=\alpha_k-\beta_k$ such that $\{\alpha_k\}$ and $\{\beta_k\}$ are non-negative non-increasing for $\{\epsilon_k\}_0^{\infty}\in BV$ [see Hirchman 1962, pp. 101, Prob. 8&9].

$$\text{Now}\quad\left|\sum_a^b q_k\epsilon_k e^{i(n-k)t}\right|=\left|\sum_a^b q_k\alpha_k e^{i(n-k)t}-\sum_a^b q_k\beta_k e^{i(n-k)t}\right|$$

$$\leqslant\alpha_a\max_{a\leqslant b'\leqslant b}\left|\sum_a^{b''}q_k e^{i(n-k)t}\right|+\beta_a\max_{a\leqslant b''\leqslant b}\left|\sum_a^{b'}q_k e^{i(n-k)t}\right|$$

$$\leqslant C_1Q_\tau+C_2Q_\tau=CQ_\tau \text{ (see McFadden 1942).}$$

§ 3. *Proof of Theorem* 1—We have

$$\varphi_1(t)\in BV(0,\pi)\Rightarrow\sum_{n=0}^{\infty}A_n(t)\text{ converges;}$$

and when $\{\epsilon_n\} \in BV(0,\infty)$, $\sum_{n=0}^{\infty} A_n(t)$ converges $\Rightarrow \sum_{n=0}^{\infty} \epsilon_n A_n(t) < \infty$.

So by lemma 1 it is sufficient to prove that $\sum_{n=1}^{\infty} \frac{|\sigma_n|}{(n+1)P_{n-1}} < \infty$ ...(3.1)

where $\sigma_n = \sum_{k=1}^{n} p_{n-k}\, k\epsilon_k A_k(x)$.

Now $A_n(x) = 2/\pi \int_0^{\pi} \varphi(t) \cos nt\, dt = -(2/\pi) \int_0^{\pi} \frac{\sin nt}{n}\, d\varphi_1(t) + (2/\pi) \int_0^{\pi} t \cos nt\, d\varphi_1(t)$

and hence to prove (3.1) we show that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \int_0^{\pi} d\varphi_1(t) \sum_{\nu=1}^{n} p_{n-\nu}\, \epsilon_\nu \sin \nu t \right| < \infty \qquad \text{...(3.2)}$$

and $$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \int_0^{\pi} t\, d\varphi_1(t) \sum_{\nu=1}^{n} p_{n-\nu}\, \nu\epsilon_\nu \cos \nu t \right| < \infty. \qquad \text{..(3.3)}$$

But $\int_0^{\pi} |d\varphi_1(t)| < \infty$, so by mean value theorem it is sufficient to show that

$$I_1 = \sum^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=1}^{n} p_{n-\nu}\, \epsilon_\nu \sin \nu t \right| = O(1) \qquad \text{...(3.4)}$$

and $$I_2 = t \sum^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=1}^{n} p_{n-\nu}\, \epsilon_\nu\, \nu \cos \nu t \right| = O(1) \qquad \text{...(3.5)}$$

uniformly for $0 < t \leqslant \pi$,

Let $I_1 = \sum_{n \leqslant 1/t} + \sum_{n > 1/t}$ so that,

$$\sum_{n \leqslant 1/t} \leqslant \sum_{n \leqslant 1/t} \frac{1}{(n+1)P_{n-1}} \sum_{\nu=1}^{n} |p_{n-\nu}|\, \nu t \leqslant \sum_{n \leqslant 1/t} \frac{ntP_n}{(n+1)P_{n-1}} \leqslant C$$

and $$\sum_{n > 1/t} = c_1 \sum_{n > 1/t} \frac{p_{n-1} t^{-1}}{(n+1)P_{n-1}}, \left(\{P_{n-\nu}\} \searrow \text{ with respect to } \nu \right)$$

$$< C\, t^{-1} \sum_{n > 1/t} 1/n < C, \left(\frac{p_n}{P_{n-1}} \leqslant \frac{C}{n+1} \text{ by hyp. (ii)} \right)$$

(equation continned on p. 462)

$$I_2 = t\sum_{n\leqslant 1/t} + t\sum_{n>1/t} = J_1 + J_2 \text{ (say)},$$

where $J_1 \leqslant Ct\sum_{n\leqslant 1/t} \frac{1}{(n+1)P_{n-1}}\, p_{n-1} \sum_{\nu=1}^{n} \nu \leqslant Ct\sum_{n\leqslant 1/t} \frac{np_n}{P_{n-1}} < C.$

Also $J_2 = t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=1}^{n} p_{n-\nu}\epsilon_\nu \, \nu \cos \nu t \right|$

$$= t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=0}^{n} D_\nu(t) \triangle (p_{n-\nu}\nu\epsilon_\nu) \right|$$

(Δ operator acts on ν so that $\Delta f_\nu = f_\nu - f_{\nu+1}$)

$$= t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=0}^{n} \left[D_\nu(t)\nu\epsilon_\nu \Delta p_{n-\nu} + D_\nu(t)\, p_{n-\nu-1}\nu\Delta\epsilon_\nu - D_\nu(t) p_{n-\nu-1}\epsilon_{\nu+1} \right] \right|$$

$$\leqslant \Sigma_1 + \Sigma_2 + \Sigma_3 \text{ (say)},$$

where $\sum_1 = t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=0}^{n} D_\nu(t)\nu\, \epsilon_\nu \, \Delta p_{n-\nu} \right|$

Let k be a constant so chosen that $\{p_n - p_{n-1}\}$ is monotonic for $n > k$. Then

$$\sum_{\nu=1}^{n-k-1} | p_{n-\nu} \quad p_{n-\nu-1} | = \sum_{\nu=1}^{n-k-1} (p_{n-\nu} - p_{n-\nu-1}) \;;\; p_{n-\nu} - p_{n-\nu-1} \geqslant 0, \qquad \ldots(3.6)$$

$= \begin{cases} O(n+1)\,(p_n - p_{n-1}) & \text{(for } \{p_{n-\nu} - p_{n-\nu-1}\} \text{ monotonic non-increasing with respect to } \nu) \\ O(n+1) & \text{(for } \{p_{n-\nu} - p_{n-\nu-1}\} \text{ monotonic non-decreasing with respect to } \nu). \end{cases}$

Now $\sum_1 \leqslant t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=n-k}^{n} (p_{n-\nu} - p_{n-\nu-1})\, \nu\epsilon_\nu \, D_\nu(t) \right|$

$$+ t\sum_{n>1/t} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=1}^{n-k-1} (p_{n-\nu} - p_{n-\nu-1})\, \nu\epsilon_\nu \, D_\nu(t) \right|$$

$$= K_{2\cdot 1} + K_{2\cdot 2} \text{ (say)},$$

But $K_{2\cdot 1} = O(1) \sum_{n>1/t} 1/P_{n-1} \sum_{\nu=n-k}^{n} | p_{n-\nu} - p_{n-\nu-1} |$

$$= O(1) \sum_{n>1/t} \frac{1}{P_{n-1}} \text{ (since } k \text{ is fixed)}$$

$= O(1)$ (by hypothesis (iii)).

Now in $K_{2\cdot 2}$ let $\{p_{n-\nu}-p_{n-\nu-1}\}$ is monotonic non-increasing with respect to ν, so that $\{p_r-p_{r-1}\}$ is monotonic non-decreasing for $k+1\leqslant r\leqslant n$. Hence

$$K_{2\cdot 2}=t\sum_{n<1/t}\frac{1}{(n+1)P_{n-1}}\left|\sum_{r=k+1}^{n-1}(p_r-p_{r-1})(n-r)\,\epsilon_{n-r}\,D_{n-r}(t)\right|$$

$$= t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}(p_{n-1}-p_{n-2})(n-k-1)\max_{k+1\leqslant a\leqslant b\leqslant n-1}\left|\sum_{r=a}^{b}\epsilon_{n-1}D_{n-r}(t)\right|$$

$$= O(t^{-1})\sum_{n>1/t}\frac{(p_n-p_{n-1})}{P_{n-1}}\ \text{(by Lemma 2)},$$

$$= O(t^{-1})\sum_{n>1/t}\left[\frac{1}{n}\left\{\frac{(n+1)p_n}{P_n}-\frac{np_{n-1}}{P_{n-1}}\right\}+O(1/n^2)\right]$$

$$= O(t^{-1})\sum_{n>1/t}1/n\left|\Delta\left\{\frac{(n+1)p_n}{P_n}\right\}\right|+O(t^{-1})\sum_{n>1/t}1/n^2$$

$= O(1)$ [by hypothesis (ii)].

Again in a similar way, let $\{p_{n-\nu}-p_{n-\nu+1}\}$ is monotonic non-decreasing so that $\{p_r-p_{r-1}\}\equiv\{-\Delta p_r\}$ is monotonic non-increasing. Hence

$$K_{2\cdot 2} = t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}\left|\sum_{r=k+1}^{n-1}(p_r-p_{r-1})(n-r)\,\epsilon_{n-r}D_{n-r}(t)\right|$$

$$= t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}(n-k-1)\max_{k+1\leqslant a\leqslant n-1}\left|\sum_{r=a}^{n-1}\Delta p_r\epsilon_{n-r}D_{n-r}(t)\right|$$

$$= O(1)\sum_{n>1/t}\frac{p_\tau}{P_{n-1}}\ \text{(by Lemma 2)}$$

$$=O(p_\tau)\frac{C\tau}{P_\tau}\qquad =O\left(\frac{\tau p_\tau}{P_\tau}\right) = O(1)\ \text{(by hypothesis (ii)},$$

so that $\Sigma_1 = O(1)$ uniformly in $0<t\leqslant\pi$.

$$\sum\nolimits_3 = t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}\left|\sum_{\nu=0}^{n}D_\nu(t)\,p_{n-\nu-1}\,\epsilon_{\nu+1}\right|$$

$$= t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}\left|\sum_{\nu=0}^{n}(p_{n-\nu-1}-p_{n-\nu-2})\sum_{\mu=0}^{\nu}\epsilon_{\mu+1}D_\mu(t)\right|$$

(*equation continued on page* 464)

$$= t\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}O(1/t^2)\sum_{\nu=0}^{n}\left|(p_{n-\nu-1}-p_{n-\nu-2})\right|$$

$$=O(t^{-1})\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}\sum_{\nu=0}^{n}(p_{n-\nu-1}-p_{n-\nu-2}) = O(t^{-1})\sum_{n>1/t}\frac{p_n}{(n+1)P_{n-1}}$$

$$=O(t^{-1})\sum_{n>1/t}1/n^2 = O(1), \quad \text{uniformly in } t\in(0,\pi).$$

Also $\sum_2 = O\left\{\sum_{n>1/t}\frac{1}{(n+1)P_{n-1}}\sum_{\nu=0}^{n-1}p_{n-\nu-1}\,\nu\,|\Delta\epsilon_\nu|\right\}$

$$=O\left\{\sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\sum_{\nu=0}^{n-1}p_{n-\nu-1}\,\nu\,|\Delta\epsilon_\nu|\right\}$$

$$=O\left\{\sum_{\nu=1}^{\infty}\nu\,|\Delta\epsilon_\nu|\sum_{n=\nu+1}^{\infty}\frac{p_{n-\nu-1}}{(n+1)P_{n-1}}\right\}$$

But

$$\sum_{n=\nu+1}^{2\nu}\frac{p_{n-\nu-1}}{(n+1)P_{n-1}} \leqslant \frac{1}{(\nu+2)P_\nu}\sum_{n=\nu+1}^{2\nu}p_{n-\nu-1} \leqslant \frac{P_{\nu-\nu}}{(\nu+2)P_\nu} < \frac{1}{\nu+2}$$

and by hypothesis (ii) $\dfrac{(n+1)p_n}{P_n} \leqslant M$ (M a constant),

we have $\quad p_{n-\nu-1} < \dfrac{MP_{n-\nu-1}}{n-\nu} < \dfrac{MP_{n-1}}{n-\nu}$

and so $\quad \displaystyle\sum_{n=2\nu+1}^{\infty}\frac{p_{n-\nu-1}}{(n+1)P_{n-1}} < M\sum_{n=2\nu+1}^{\infty}\frac{1}{(n+1)(n-\nu)} = O(1/\nu)$

i.e. $\quad \displaystyle\sum_{n=\nu+1}^{\infty}\frac{p_{n-\nu-1}}{(n+1)P_{n-1}} = O\left(\frac{1}{\nu}\right)$

We, therefore, have $\displaystyle\sum_2 = O\left\{\sum_{n=1}^{\infty}|\Delta\epsilon_\nu|\right\} = O(1)$

Collecting the orders of Σ_1, Σ_2 and Σ_3 we have

$$I_2 = O(1).$$

This completes the proof of Theorem 1.

Remark 2: If in addition to $\{\epsilon_\nu\} \in BV(0,\infty)$, we also assume $\{\nu\Delta\epsilon_\nu\} \in BV(0,\infty)$ in Theorem 1, then the proof of our theorem is exactly the same as that of Bhatt (1968); and our Corollaries 1.2 are still true.

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