Indian J. pure appl. Math., **13**(4) : 466-470, April 1982

ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE CLASS Lip (α, p)

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(*Received* 16 *February* 1981; *after revision* 6 *July* 1981)

In this paper the author has obtained the degree of approximation of certain functions belonging to the class Lip (α, p) taking the concept of 'almost convergence' introduced by Lorentz (1948).

§ 1. Let f be a 2π periodic function integrable L^p ($p > 1$) and let

$$f \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \qquad \text{...(1.1)}$$

be its Fourier series.

A function $f \in$ Lip α if

$$f(x+h) - f(x) = O(|h|^{\alpha}) \text{ for } 0 < \alpha \leqslant 1, \qquad \text{...(1.2)}$$

We define the norm $\|\cdot\|_p$ by

$$\|f\|_p = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \; p \geqslant 1 \qquad \text{...(1.3)}$$

and the degree of approximation $E_n(f)$ by

$$E_n(f) = \min_{T_n} \|f - T_n\|_p \qquad \text{...(1.4)}$$

where $T_n(x)$ is a trigonometrical Polynomial of degree n.

We say that $f \in$ Lip (α, q) for $a \leqslant x \leqslant b$ if

$$\left\{ \int_a^b |f(x+h) - f(x)|^q \, dx \right\}^{1/q} \leqslant A\,|h|^{\alpha}, \; 0 < \alpha \leqslant 1, q \geqslant 1, \qquad \text{...(1.5)}$$

where A is some constant. [see Def. 5.38 of McFadden (1942)].

We write $\phi(t) = [f(x+t) + f(x-t) - 2f(x)]$.

Lorentz (1948) has defined:

Definition L_1—A sequence $\{s_n\}$ is said to almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} S_k = S, \qquad \text{...(1.6)}$$

uniformly with respect to p.

Recently Sharma *et al.* (1977) have defined almost Borel summability.

Sharma and Qureshi (1980) defined:

Definition L_2—A series Σu_n with the sequence of partial sums $\{s_n\}$ is said to be almost Riesz summable to S, provided

$$t_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_{k,p} \longrightarrow S \text{ as } n \longrightarrow \infty \qquad (\ldots 1.7)$$

uniformly with respect to p, where (Sharma *et al.* 1977)

$$S_{k,p} = \frac{1}{k+1} \sum_{\mu=p}^{k+p} S_\mu$$

and $\{p_n\}$ be a sequence of non-negative constants, such that $p_0 > 0$ and $P_n = p_0 + p_1 + p_2 + \ldots + p_n$.

The Riesz means are regular if and only if $P_n \to \infty$ with n [see Theorem 1.4.4 of Petersen (1966)].

§ 2. Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the (N, p_n) method such that

$$P_n \equiv P(n) = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty. \qquad \ldots(2.1)$$

We write $p(y) = p_{[y]}$ and $P(y) = P_{[y]}$,

where $[y]$ as usual denotes the greatest integer less than y.

Sahney and Rao (1972 a, b) have proved the following theorem.

Theorem A—If f is periodic and belongs to the class Lip (α, p), $0 < \alpha \leqslant 1$ and let $\{p_n\}$ be defined as in (2.1) and

$$\left(\int_1^n \frac{(P(y))^q}{y^{q\alpha+2-q}}\, dy \right)^{1/q} = O\left(\frac{P(n)}{n^{(q\alpha-1/q-1)/q}} \right)$$

then $$\| f - t_n \|_p = O\left(\frac{1}{n^{(\alpha p - 1/p)}} \right)$$

where $$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k.$$

i.e., the (N, p_n) mean of the Fourier series (1.1).

Sharma and Qureshi (1980) have proved the following theorem:

Theorem B—The degree of approximation of a periodic function f with period 2π and belonging to the class of Lip α by almost Riesz means of its Fourier series is given by

$$\max_{0 \leqslant x \leqslant 2\pi} \left| f(x) - t_{n,p}(x) \right| = \begin{cases} O\left\{ \left(\dfrac{p_n}{P_n} \right)^\alpha \right\}; & 0 < \alpha < 1 \\ O\left\{ \dfrac{p_n}{P_n} \log \dfrac{P_n}{p_n} \right\}; & \alpha = 1 \end{cases}$$

where Riesz means are regular and $0 < p_n \uparrow$ with $n > n_0$.

Our object of this paper is to prove the following theorem.

Theorem C—The degree of approximation of a periodic function f belonging to the class Lip (α, p) for $0 < \alpha \leqslant 1$ by almost Riesz means is given by

$$\| f - t_{n,p} \|_p = O\left\{ \left(\frac{p_n}{P_n}\right)^{\alpha - \frac{1}{p}} \right\}$$

where Riesz means are regular and $0 < p_n \uparrow$ with $n \geqslant n_0$.

To prove the theorem we shall need the following lemma.

Lemma (Lemma 5.40 of McFadden 1942)—If f belongs to Lip (α, q) on $[0, \pi]$. then $\phi(t)$ also belongs to Lip (α, q) on $[0, \pi]$.

§ 3. *Proof of the theorem*—Following Sharma *et al.* (1977), we write

$$S_{k,p}(x) - f(x) = \frac{1}{2\pi(k+1)} \int_0^{\pi} \phi(t) \frac{[\cos pt - \cos (k+p+1)t]}{2 \sin^2 t/_2} \, dt.$$

We have $f(t) - t_{n,p}(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \{ f(t) - S_{k,p}(t) \}$

$$= \frac{1}{2\pi P_n} \int_0^{\pi} \phi(t) \sum_{k=0}^{n} \frac{p_k}{k+1} \frac{[\cos (k+p+1) t - \cos pt]}{2 \sin^2 t/_2} \, dt$$

$$= \frac{1}{2\pi P_n} \left[\int_0^{\frac{p_n}{P_n}} + \int_{\frac{p_n}{P_n}}^{\pi} \right] \phi(t) \left(\sum_{k=0}^{n} - \frac{p_k}{k+1} \frac{\sin (k+2p+1)t/_2 \sin (k+1)t/_2}{\sin^2 t/_2} \right) dt$$

$= I_1 + I_2$, say.

Now $$I_1 = \frac{1}{2\pi P_n} \int_0^{\frac{p_n}{P_n}} \phi(t) \sum_{k=0}^{n} - \frac{p_k}{k+1} \frac{\sin (k+2p+1) t/_2 \sin (k+1) t/_2}{\sin^2 t/_2} \, dt.$$

By Hölders inequality and the lemma, we have

$$I_1 \leqslant \frac{1}{2\pi P_n} \left\{ \left(\int_0^{\frac{p_n}{P_n}} \left(\frac{t \,|\phi(t)|}{t^{\alpha}} \right)^p dt \right)^{1/p} \right\} \times$$

$$\left\{ \left(\int_0^{\frac{p_n}{P_n}} \left(\frac{1}{t^{1-\alpha}} \left| \sum_{k=0}^{n} \frac{p_k \sin (k+2p+1)t/_2 \sin (k+1)t/_2}{k+1 \quad\quad \sin^2 t/_2} \right| \right)^q dt \right)^{1/q} \right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left(\frac{p_n}{P_n}\right) O\left\{ \left(\int_0^{\frac{p_n}{P_n}} \left(\frac{1}{t^{1-\alpha}} \sum_{k=0}^{n} \frac{p_k (k+1)}{k+1} \cdot \frac{1}{t} \right)^q dt \right)^{1/q} \right\}$$

(*equation continued on p.* 469)

$$= O\left(\frac{p_n}{P_n}\right) O\left\{\left(\int_0^{\frac{p_n}{P_n}} t^{\alpha q-2q}\, dt\right)^{1/q}\right\} = O\left(\frac{p_n}{P_n}\right) O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-2+1/q}\right\}$$

$$= O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-1+\frac{1}{q}}\right\} = O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-\frac{1}{p}}\right\}$$

where $1/p + 1/q = 1$ such that $1 \leqslant p \leqslant \infty$.

Also, similarly, as above

$$I_2 = O\left(\frac{1}{P_n}\right)\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi} \left(\frac{t^{-\delta}\,|\,\phi(t)\,|}{t^{\alpha}}\right)^p dt\right)^{\frac{1}{p}}\right\} \times$$

$$\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi} \left|\frac{1}{t^{-\delta-\alpha}}\sum_{k=0}^{n}\frac{p_k}{k+1}\;\frac{\sin(k+2p+1)t/_2\,\sin(k+1)t/_2}{\sin^2 t/_2}\right|^q dt\right)^{\frac{1}{q}}\right\},$$

where δ is any finite quantity-

Now $$I_2 = O\left(\frac{1}{P_n}\right) O\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi}\left(\frac{t^{-\delta}\, t^{\alpha-1/p}}{t^{\alpha}}\right)^p dt\right)^{1/p}\right\} \times$$

$$\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi}\left(\frac{1}{t^{-\delta-\alpha}}\sum_{k=0}^{n}\frac{|\,p_k\sin(k+2p+1)\,t/_2\,|}{(k+1)}\;\frac{(k+1)\,|\,\sin t/_2\,|}{\sin^2 t/_2}\right)^q dt\right)^{1/q}\right\}$$

$$= O\left(\frac{1}{P_n}\right) O\left\{\left(\frac{p_n}{P_n}\right)^{-\delta}\right\} O\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi}\left(\frac{t^{\alpha+\delta}}{\sin t/_2}\sum_{k=0}^{n}\left|\,p_k\sin(k+2p+1)\frac{t}{2}\right|\right)^q dt\right)^{\frac{1}{q}}\right\}$$

$$= O\left(\frac{p_n}{P_n}\right) O\left\{\left(\frac{p_n}{P_n}\right)^{-\delta}\right\} O\left\{\left(\int_{\frac{p_n}{P_n}}^{\pi} t^{\alpha q+\delta q-2q}\, dt\right)^{\frac{1}{q}}\right\}$$

since $\{p_n\}$ is monotonic increasing, we have

$$\sum_{k=0}^{n} p_k \sin(k+2p+1)\, t/_2 \leqslant p_n \sum_{k=0}^{n} \sin(k+2p+1)\, t/_2$$

$$= O\left(\frac{p_n}{t}\right) \qquad \text{[alternatively see (1],]}$$

Now $I_2 = O\left(\frac{p_n}{P_n}\right) O\left\{\left(\frac{p_n}{P_n}\right)^{-\delta} O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha+\delta-2+\frac{1}{q}}\right\}\right.$

$$= O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-1+\frac{1}{q}}\right\} = O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-\frac{1}{p}}\right\}.$$

Hence $\|f - t_{n,p}\|_p = O\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-\frac{1}{p}}\right\}.$

This completes the proof of Theorem C.

Remark : It is to be noted that as $p \to \infty$ (and therefore $q=1$), Theorem C is equivalent to Theorem B.

ACKNOWLEDGEMENT

The author is thankful to Dr P. L. Sharma, Professor and Head of the Mathematics department, University of Saugar, for his keen interest and valuable guidance in the preparation of the paper. He is also thankful to the referee for his kind suggestions for the improvement of the paper.

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