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ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Let $P_\alpha(A, B)$ denote the class of functions $f(z) = z + a_2z^2 + \ldots$ which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy the condition

$$\frac{f(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad \text{where} \quad -1 \leqslant B < A \leqslant 1,$$

$g(z)$ being starlike of order α in $E(0 \leqslant \alpha < 1)$ and $w(z)$ is regular in E with $w(0) = 0$, $|w(z)| < 1$ in E. By $Q_\alpha(A, B)$ we denote the class of functions $F(z) = z^{-1} + C_0 + C_1z + C_2z^2 + \ldots$ regular in $0 < |z| < 1$ satisfying the condition $\frac{F(z)}{G(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$ with $-1 \leqslant B < A \leqslant 1$ and $w(z)$ regular in E, $w(0) = 0$ and $|w(z)| < 1$. Here $G(z) = z^{-1} + d_0 + d_1z + \ldots$ is regular and starlike of order α in $0 < |z| < 1$ i.e.

$$-\operatorname{Re} z\frac{G'(z)}{G(z)} > \alpha \text{ in } 0 < |z| < 1.$$

In this paper we obtain the distortion theorem, coefficient estimates and the radius of starlikeness of the class $P_\alpha(A, B)$. The radius of starlikeness of the class $Q_\alpha(A, B)$ has also been obtained. The results of Goel and Sohi (1980) could be obtained as particular cases of our results.

1. INTRODUCTION

In a recent paper Goel and Sohi (1980) have obtained distortion theorem and coefficient estimates for the class of functions $S_\lambda(\alpha, \beta)$ consisting of all those functions $f(z) = z + a_2z^2 + \ldots$ which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition

$$\frac{|[f(z)/g(z)] - 1|}{|[\lambda f(z)/g(z)] - 1|} < \beta, \quad Z \in E, \quad 0 \leqslant \lambda < 1 \qquad \ldots(1)$$

$0 < \beta \leqslant 1$ where $g(z) = z + b_2z^2 + \ldots$ is regular and starlike of order $\alpha(0 \leqslant \alpha < 1)$ in E. They have also studied the class $K_\lambda(\alpha, \beta)$ consisting of those functions $F(z) = z^{-1} + C_0 + C_1z + C_2z^2 + \ldots$ which are regular in $0 < |z| < 1$ and satisfying the condition

$$\frac{|[F(z)/G(z)] - 1|}{|[\lambda F(z)/G(z)] - 1|} < \beta, \quad 0 < |z| < 1, \quad 0 < \beta \leqslant 1 \qquad \ldots(2)$$

where $G(z)=z^{-1}+d_0+d_1z+\ldots$ is regular and starlike of order α in $0<|z|<1$. If $f(z)\in S_\lambda(\alpha,\beta)$, it can be represented in the form

$$\frac{f(z)}{g(z)}=\frac{1-\beta w(z)}{1+\lambda\beta w(z)} \qquad \ldots(3)$$

where $w(z)$ is analytic in E and satisfies the conditions $|w(z)|<1$ and $w(0)=0$. A similar expression for $F(z)/G(z)$ holds if $F(z)\in K_\lambda(\alpha,\beta)$. The radii of starlikeness have also been obtained for the classes $S_\lambda(\alpha,\beta)$ and $K_\lambda(\alpha,\beta)$. It is our aim in this paper to generalize these results to the classes $P_\alpha(A,B)$ and $Q_\alpha(A,B)$.

Definition 1 — $P_\alpha(A,B)$ contains the functions $f(z)=z+a_2z^2+\ldots$ regular in the unit disc $E=\{z:|z|<1\}$ and satisfying the condition

$$\frac{f(z)}{g(z)}=\frac{1+Aw(z)}{1+Bw(z)}, \qquad z\in E \qquad \ldots(4)$$

where $-1\leqslant B<A\leqslant 1$ and $w(z)$ is regular in E with $w(0)=0$, $|w(z)|<1$, in E. Here $g(z)$ is a starlike function of order α in E i.e. $\mathrm{Re}\,\frac{zg'(z)}{g(z)}>\alpha$, $z\in E$.

Definition 2 — $Q_\alpha(A,B)$ denotes the class of functions

$$F(z)=z^{-1}+C_0+c_1z+c_2z^2+\ldots$$

which are regular in $0<|z|<1$ and satisfy the condition

$$\frac{F(z)}{G(z)}=\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}^{-1} \quad \text{where} \quad -1\leqslant B<A\leqslant 1 \qquad \ldots(5)$$

and $w(z)$ is regular in E and satisfies $|w(z)|<1$ and $w(0)=1$. Here

$$G(z)=z^{-1}+d_0+d_1z+d_2z^2\ldots$$

is starlike of order α and regular in $0<|z|<1$

i.e. $$-\mathrm{Re}\,\frac{zG'(z)}{G(z)}>\alpha, \quad 0<|z|<1.$$

2. Distortion Theorem for the Class $P_\alpha(A, B)$

Theorem 1 — If $f(z)\in P_\alpha(A,B)$ then for $|z|=r$, $0\leqslant r<1$,

$$\frac{1-Ar}{1-Br}\,\frac{r}{(1+r)^{2-2\alpha}}\leqslant 1\,f(z)\,|\leqslant\frac{1+Ar}{1+Br}\,\frac{r}{(1-r)^{2-2\alpha}}. \qquad \ldots(6)$$

The result is sharp.

Proof: Since $f(z)\in P_\alpha(A,B)$ we have by definition,

$$\frac{f(z)}{g(z)}=\frac{1+Aw(z)}{1+Bw(z)}, \quad -1\leqslant B<A\leqslant 1$$

by Schwarz's lemma we have $|w(z)| \leqslant |z|$.

It is well known that the images of closed disc $|z| \leqslant r$ under all the transformation $w = P(z)$ where $P(z) = \frac{1 + A(w)(z)}{1 + Bw(z)}$ with $w(z)$ regular in E, $w(0) = 0$ and $|w(z)| < 1$ in E, are contained in the closed disc with the centre C and the radius P where

$$C = \frac{1 - ABr^2}{1 - B^2r^2}, \quad P = \frac{(A - B)r}{1 - B^2r^2}.$$

Therefore here we must have

$$\left| \frac{f(z)}{g(z)} - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leqslant \frac{(A - B)r}{1 - B^2r^2}. \qquad \ldots(7)$$

Thus

$$\left[\frac{1 - Ar}{1 - Br}\right] \leqslant \left| \frac{f(z)}{g(z)} \right| \leqslant \left[\frac{1 + Ar}{1 + Br}\right]. \qquad \ldots(8)$$

Further $g(z)$ is starlike of order α and therefore we have (Pinchuk 1968)

$$\frac{r}{(1 + r)^{2-2\alpha}} \leqslant |g(z)| \leqslant \frac{r}{(1 - r)^{2-2\alpha}} \quad |z| = r < 1. \qquad \ldots(9)$$

(8) and (9) together imply the inequality (6). This result is sharp. Take $f(z) = \frac{1 + Az}{1 + Bz}$ and $g(z) = \frac{z}{(1 - \theta z)^{2-2\alpha}}$, $|\theta| = 1$. Obviously $f(z) \in P_\alpha(A, B)$ and the equality sign occurs in (6).

Remarks: On taking $B = -\lambda\beta$ and $A = \beta$ with $w(z)$ replaced by $-w(z)$ we get the results of Goel and Sohi (1980).

3. Coefficient Estimates for the Class $P_\alpha(A, B)$

Theorem 2 — If $f(z) \in P_\alpha(A, B)$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

(i) $|a_2| \leqslant 2(1 - \alpha) + (A - B)$

(ii) $|a_3| \leqslant (1 - \alpha)(3 - 2\alpha) + 2(1 - \alpha)(A - B) + (A - B)$ the estimate (i) is sharp.

Proof: Let $\frac{f(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$. Then

$$w(z) = \frac{f(z) - g(z)}{Ag(z) - Bf(z)}. \qquad \ldots(10)$$

Let the power series for $w(z)$ be given by

$$w(z) = \sum_{n=1}^{\infty} w_n z^n. \qquad ...(11)$$

On substituting the power series for $f(z)$, $g(z)$ and $w(z)$ in (10) we get

$$[A(z + \sum_{m=2}^{\infty} b_m z^m) - B(z + \sum_{2}^{\infty} a_m z^m)] (\sum_{1}^{\infty} w_m z^m)$$

$$= \sum_{2}^{\infty} (a_m - b_m) z^m. \qquad ...(12)$$

Equating the coefficients of Z^2 and Z^3 on both sides of (12) we get

$$(A - B)w_1 = (a_2 - b_2) \qquad ...(13)$$

$$(A - B)w_2 + (Ab_2 - Ba_2) w_1 = (a_3 - b_3). \qquad ...(14)$$

We can now use the following results:

$$| b_2 | \leqslant 2 - 2\alpha, \quad | b_3 | \leqslant \frac{(2 - 2\alpha)(3 - 2\alpha)}{2!} \text{ (Robertson 1936)} \qquad ...(15)$$

and

$$| w_1 | \leqslant 1 \text{ and } | w_2 | \leqslant 1 - | w_1 |^2. \qquad ...(16)$$

(13) and (14) now give the inequalities (i) and (ii) in the theorem. The bound is sharp in (i) for $f(z) = \dfrac{z + Az^2}{(1 - z)^{2-2\alpha} (1 + Bz)}$.

Remarks : 1. Clearly the corresponding results in Goel and Sohi (1980) are obtainable by putting $A = \beta$, $B = -\lambda\beta$ and replacing $w(z)$ by $-w(z)$.

2. When $B = 0$ we get $f(z)/g(z) = 1 + \beta w(z)$ and if $g(z)$ is a starlike function in E, the inequalities $| a_n | \leqslant \beta(n - 1) + n, n \geqslant 2$ with sharp bounds as discussed in Goel and Sohi (1980) are obtainable here.

4. Argument of $f(z)/z$ when $f(z) \in P_\alpha(A, B)$

Theorem 3 — If $f(z) \in P_\alpha(A, B)$, then

$$\left| \arg \frac{f(z)}{z} \right| \leqslant 2(1 - \alpha) \sin^{-1} r + \sin^{-1} \frac{(A - B)r}{1 - ABr^2}, \quad | z | = r$$

The bound is sharp.

Proof : $f(z) \in P_\alpha(A, B)$ implies that $\dfrac{f(z)}{g(z)} = \dfrac{1 + Aw(z)}{1 + Bw(z)}$, $-1 \leqslant B < A \leqslant 1$, $w(z)$ regular in E with $w(0) = 1$ and $| w(z) | < 1$.

Now $\left| \frac{f(z)}{g(z)} - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leqslant \frac{(A - B)r}{1 - B^2r^2}$. This implies clearly that

$$\left| \arg \frac{f(z)}{g(z)} \right| \leqslant \sin^{-1} \frac{(A - B)r}{1 - ABr^2} . \quad \ldots(17)$$

Since g is starlike of order α we have (Pinchuk 1968)

$$\left| \arg \frac{g(z)}{z} \right| \leqslant 2(1 - \alpha) \sin^{-1} r. \quad \ldots(18)$$

Using (17) and (18) we get, $\left| \arg \frac{f(z)}{z} \right| \leqslant 2(1 - \alpha) \sin^{-1} r + \sin^{-1} \frac{(A - B)r}{1 - ABr^2}$ as required.

Sharpness follows by taking

$$\frac{f(z)}{g(z)} = \frac{1 + A\,\theta_1 z}{1 + B\,\theta_1 z}, \quad |\theta_1| = 1 \quad \ldots(19)$$

and

$$g(z) = \frac{z}{(1 + \theta_2 z)^{-2+2\alpha}}, \quad |\theta_2| = 1 \quad \ldots(20)$$

where

$$\theta_1 = \left[\frac{r}{z} \left\{ \frac{-(A + B)r}{1 + ABr^2} + \frac{i(1 - A^2r^2)^{1/2}(1 - B^2r^2)^{1/2}}{1 + ABr^2} \right\} \right]$$

at any point on $|z| = r$

and we get

$$\arg \frac{f(z)}{g(z)} = \sin^{-1} \frac{(A - B)r}{(1 - ABr^2)} \quad \ldots(21)$$

for this choice. Taking

$$\theta_2 = \frac{r}{z} [-r + i\sqrt{1 - r^2}] \text{ at any point on } |z| = r \text{ in (20)}$$

we get

$$\arg \frac{g(z)}{z} = \arg (1 + \theta_2 z)^{2-2\alpha}$$

$$= (2 - 2\alpha) \sin^{-1} r \quad \ldots(22)$$

(21) and (22) give together for this choice of $f(z)$ and $g(z)$, θ_1 and θ_2, as described above,

$$\arg \frac{f(z)}{z} = (2 - 2\alpha) \sin^{-1} r + \sin^{-1} \frac{(A - B)r}{1 - ABr^2} \text{ on } |z| = r.$$

Thus the extremal function is

$$f(z) = \left(\frac{1 + A\theta_1 z}{1 + B\theta_1 z}\right)\{(z1 + \theta_2 z)^{2-2\alpha}\}$$

at all points on $|z| = r$.

Remarks : Again on putting $A = \beta > 0$ and $B = -\lambda\beta$ and replacing $w(z)$ by $-w(z)$ in our result, we get the corresponding results of Goel and Sohi (1980).

5. Radius of Starlikeness for $P_\alpha(A, B)$

The following results will be used (Anh and Tuan 1979).

Lemma — If $p(z) \in P(A, B)$ i.e. $p(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}$ where $-1 \leqslant B < A \leqslant 1$ and $w(z)$ is regular in E with $w(0) = 0$ and $|w(z)| < 1$ in E, then on $|z| = r < 1$

$$\text{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \frac{-(A-B)r}{(1-Ar)(1-Br)}, \quad R_1 \leqslant R_2$$

$$\geqslant \frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}[(L_1K_1)^{1/2} - (1 - ABr^2)] \text{ if } R_2 \leqslant R_1$$

where $R_1 = (L_1/K_1)^{1/2}$, $R_2 = \dfrac{1 - Ar}{1 - Br}$, $L_1 = (1 - A)(1 + Ar^2)$, $K_1 = (1 - B) \times (1 + Br^2)$.

The result is sharp.

Remarks : Lemma 1 is a particular case of Theorem 1 of Anh and Tuan (1979) obtained by putting $\alpha = 0$ and $\beta = 1$.

Theorem 4 — If $f(z) \in P_\alpha(A, B)$ then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \begin{cases} M_1(r) & \text{for } R_1 \leqslant R_2 \\ M_2(r) & \text{for } R_2 \leqslant R_1 \end{cases}$$

where
$$M_1(r) = \frac{1 + (2\alpha - 1)r}{1 + r} - \frac{(A - B)r}{(1 - Ar)(1 - Br)}$$

$$M_2(r) = \frac{1 + (2\alpha - 1)r}{1 + r} + \frac{A + B}{A - B} + \frac{2[(L_1K_1)^{1/2} - (1 - ABr^2)]}{(A - B)(1 - r^2)}$$

with R_1, R_2. L_1, K_1 are as defined in the Lemma above.

Proof : $f(z) \in P_\alpha(A, B) \Rightarrow \dfrac{f(z)}{g(z)} = \dfrac{1 + Aw(z)}{1 + Bw(z)}$ and $g(z) \in S^*(\alpha)$. Now putting $p(z) = \dfrac{f(z)}{g(z)}$ in (21) we get on differentiating logarithmically,

$$\frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} - \frac{zg'(z)}{g(z)}. \qquad \text{...(22a)}$$

But using the Lemma we can get again on differentiating (21) logarithmically, and using (22a)

$$\left.\begin{aligned} \text{Re}\ \frac{zp'(z)}{p(z)} &= \text{Re}\ \frac{zf'(z)}{f(z)} - \frac{zg'(z)}{g(z)} \geqslant \frac{-(A-B)r}{(1-Ar)(1-Br)} \text{ if } R_1 \leqslant R_2 \\ &\geqslant \frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}\left[(L_1K_1)^{1/2} - (1-ABr^2)\right] \text{ if } R_2 \leqslant R_1 \end{aligned}\right\} \qquad \text{...(23)}$$

R_1, R_2, L_1, K_1 are as defined in the Lemma above. Since $g(z)$ is starlike of order α, we get

$$\text{Re}\ \frac{zg'(z)}{g(z)} \geqslant \frac{1+(2\alpha-1)r}{1+r}, \quad |z| = r. \qquad \text{...(24)}$$

Combining (23) and (24) we get the required result. Sharpness of the bounds follow if we choose $g_i(z)$ $(i = 1, 2)$ starlike of order α such that

Case i — If $R_1 \leqslant R_2$, take

$$p_1(z) = \frac{f_1(z)}{g_1(z)} = \frac{1+Az}{1+Bz} \text{ and } \frac{zg_1'(z)}{g_1(z)} = \frac{1+(2\alpha-1)z}{1+z}.$$

Then
$$\frac{zp_1'(z)}{p_1(z)} = \frac{(A-B)'z}{(1+Az)(1+Bz)}.$$

Thus at $z = -r$, $\text{Re}\left\{\frac{zp_1'(z)}{p_1(z)}\right\} = \frac{-(A-B)r}{(1-Ar)(1-Br)}$.

Case ii — if $R_2 \leqslant R_1$ take

$$p_2(z) = \frac{f_2(z)}{g_2(z)} = \frac{1+Aw_1(z)}{1+Bw_1(z)}$$

where $\frac{zg_2'(z)}{g_2(z)} = \frac{1+(2\alpha-1)w_1(z)}{1+w_1(z)}$ with $w_1(z) = \frac{z(z-c_1)}{(1-c_1z)}$ with c_1 defined by the condition

$$\text{Re}\left\{\frac{1+Aw_1(z)}{1+Bw_1(z)}\right\} = R_1 \text{ at } z = -r.$$

Now
$$\frac{zp_2'(z)}{p_2(z)} = \frac{(A-B)zw_1'(z)}{(1+Aw_1(z))(1+Bw_1(z))}.$$

In fact from the inequalities $R_2 \leqslant R_1 \leqslant c + p$ where

$$c = \frac{1 - ABr^2}{1 - B^2r^2}, \quad p = \frac{(A-B)r}{1 - B^2r^2}$$

we get $$\frac{1 - Ar}{1 - Br} \leqslant \frac{1 + AT}{1 + BT} \leqslant \frac{1 + Ar}{1 + Br}, \quad T = w_1(-r).$$

Hence $|T| \leqslant r$ and so $T^2 \leqslant r^2$ which yields

$$\frac{r^2(r + c_1)^2}{(1 + rc_1)^2} \leqslant r^2.$$

Thus $$|c_1| \leqslant 1.$$

Further $$|zw_1'(z) - w_1(z)| = \frac{|z|^2 - |w_1(z)|^2}{1 - |z|^2}$$

for $$w_1(z) = \frac{z(z - c_1)}{(1 - c_1 z)}, \quad |c_1| \leqslant 1.$$

Now from the definition of c_1 we have

$$w_1(-r) = T = \frac{1 - R_1}{BR_1 - A} = \frac{r(r + c_1)}{(1 + c_1 r)}.$$

Hence $$c_1 = \frac{r^2 - T}{r(T - 1)} \text{ and } \frac{r^2 - T^2}{(1 - r^2)} = \frac{r^2(1 - q^2)}{(1 + qr)^2}.$$

$$\therefore [zw'(z) - w(z)]_{z=-r} = \frac{r^2 - T^2}{(1 - r^2)}.$$

Now $$\text{Re}\left\{\frac{zp_2'(z)}{p_2(z)}\right\} = \text{Re}\,\frac{(A - B)\,T}{(1 + AT)(1 + BT)} - \frac{(A - B)(r^2 - T^2)}{(1 - r^2)(1 + AT)(1 + BT)}$$

$$= \frac{(A - B)}{(1 + AT)(1 + BT)}\left\{T - \frac{r^2 - T^2}{1 - r^2}\right\}.$$

Using the results $T = \dfrac{1 - R_1}{BR_1 - A}$ with

$$R_1 = \sqrt{\frac{(1 - A)(1 + Ar^2)}{(1 - B)(1 + Br^2)}} \quad \text{(Anh and Tuan 1979)}$$

and simplifying we get,

$$\left[\text{Re}\,\frac{zp_2'(z)}{p_2(z)}\right]_{z=-r}$$

$$= \frac{A + B}{A - B} + \frac{2}{(1 - r^2)(A - B)}\,[(L_1K_1)^{1/2} - (1 - ABr^2)]$$

where $L_1 = (1 - A)(1 + Ar^2)$, $K_1 = (1 - B)(1 + Br^2)$ obtained from Anh and

Tuan's result by putting $\alpha = 0$ and $\beta = 1$. Thus equality in (23) holds at $z = -r$ for

$$f_1(z) = \left(\frac{1 + Az}{1 + Bz}\right) g_1(z) \quad \text{with} \quad \frac{zg_1'(z)}{g_1(z)} = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad R_1 \leqslant R_2$$

and
$$f_2(z) = \left(\frac{1 + Aw_1(z)}{1 + Bw_1(z)}\right) g_2(z) \quad \text{with} \quad \frac{zg_2'(z)}{g_2(z)} = \frac{1 + (2\alpha - 1)w_1(z)}{1 + w_1(z)}, \quad R_2 \leqslant R_1$$

where $g_1, g_2 \in s^*(\alpha)$.

Theorem 5 — If $f(z) \in P_\alpha(A, B)$, then f is starlike in

$$|z| < \begin{cases} r_1 & \text{for} \quad R_1 \leqslant R_2 \\ r_2 & \text{for} \quad R_2 \leqslant R_1 \end{cases}$$

where R_1 and R_2 are defined as in the Lemma and r_1 and r_2 are respectively the positive roots of the following two equations:

$$(1 + (2\alpha - 1)\, r(1 - Ar)(1 - Br) - (A - B)\, r(1 + r) = 0 \qquad \ldots(25)$$

and
$$(1 + (2\alpha - 1)\, r)(A - B)(1 - r^2) + (1 + r)(1 - r^2)(A + B) + 2(1 + r)[(L_1K_1)^{1/2} - (1 - ABr^2)] = 0 \qquad \ldots(26)$$

L_1, K_1 are as defined in the Lemma above. The result is sharp.

Proof : The theorem is an immediate consequence of Theorem 4 above.

Note : The results obtained here reduce to those given in Goel and Sohi (1980) on putting $A = \beta$, $B = -\lambda\beta$ and changing $w(z)$ into $-w(z)$.

6. Radius of Starlikeness for the Class $Q_\alpha(A, B)$

Goel and Sohi (1980) have determined the radius of starlikeness for the class $K\lambda(\alpha, \beta)$. The result obtained here will generalize these results to the class $Q_\alpha(A, B)$.

Theorem 6 — If $F \in Q_\alpha(A, B)$, then for $|z| = r < 1$,

$$-\operatorname{Re} \frac{zF'(z)}{F(z)} \geqslant \begin{cases} P_1(r) & \text{for} \quad R_1 \leqslant R_2 \\ P_2(r) & \text{for} \quad R_2 \leqslant R_1 \end{cases}$$

where
$$P_1(z) = \frac{1 + (2\alpha - 1)\, r}{1 + r} - \frac{(A - B)\, r}{(1 - Ar)(A - B)}$$

$$P_2(z) = \frac{1 + (2\alpha - 1)\, r}{1 + r} + \frac{A + B}{A - B} + \frac{2}{(1 - r^2)(A - B)} \times [(L_1K_1)^{1/2} - (1 - ABr^2)].$$

PROOF : Since $F \in Q_\alpha(A, B)$ we can write

$$p(z) = \left\{\frac{F(z)}{G(z)}\right\}^{-1} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad \text{where} \quad -1 \leqslant B < A \leqslant 1 \qquad \ldots(27)$$

and $w(z)$ is regular in E with $w(0) = 0$ and $|w(z)| < 1$. Differentiating (27) logarithmically, we get

$$\frac{zp'(z)}{p(z)} = -\frac{zF'(z)}{F(z)} + \frac{zG'(z)}{G(z)}. \qquad \ldots(28)$$

$$\therefore \quad -z\frac{F'(z)}{F(z)} = -\frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}$$

$$\therefore \quad \left.\begin{aligned} -\text{Re}\left(z\frac{F'(z)}{F(z)}\right) &\geqslant -\text{Re}\frac{zG'(z)}{G(z)} - \frac{(A-B)r}{(1-Ar)(1-Br)} \text{ if } R_1 \leqslant R_2 \\ &\geqslant -\text{Re}\frac{zG'(z)}{G(z)} + \frac{A+B}{A-B} + \frac{2[(L_1K_1)^{1/2} - (1-ABr^2)]}{(A-B)(1-r^2)} \text{ if } R_2 \leqslant R_1. \end{aligned}\right\} \qquad \ldots(29)$$

Since G is starlike of order α therefore we have,

$$-\text{Re}\frac{zG'(z)}{G(z)} \geqslant \frac{1 + (2\alpha - 1)r}{1 + r}, \quad |z| = r \qquad \ldots(30)$$

(29) and (30) together give the required inequalities. The bounds are sharp. This can be seen by choosing $G_1(z)$ starlike of order α such that

$$\frac{-zG'(z)}{G(z)} = \frac{1 + (2\alpha - 1)z}{1 + z} \quad \text{if} \quad R_1 \leqslant R_2$$

$$= \frac{1 + (2\alpha - 1)w_1(z)}{1 + w_1(z)} \quad \text{if} \quad R_2 \leqslant R_1$$

and take $F_1(z)$ such that it satisfied

$$p_1(z) = \left\{\frac{F_1(z)}{G_1(z)}\right\}^{-1} = \frac{1 + Az}{1 + Bz} \quad \text{if } R_1 \leqslant R_2$$

or $$= \frac{1 + Aw_1(z)}{1 + Bw_1(z)} \quad \text{if } R_2 \leqslant R_1$$

where $w_1(z) = z(z - C_1z)/(1 - C_1z)$ with c_1 defined by the condition

$$\text{Re}\left\{\frac{1 + Aw_1(z)}{1 + Bw_1(z)}\right\} = R_1 \text{ at } z = -r.$$

Obviously $|C_1| \leqslant 1$. Here sharpness follows as in Theorem 4 at $z = -r$ on $|z| = r$.

Theorem 7 — If $F(z) \in Q_\alpha(A, B)$ then for $|z| = r < 1$ F is starlike in

(i) $0 < |z| < r_1$ for $R_1 \leqslant R_2$

(ii) $0 < |z| < r_2$ for $R_2 \leqslant R_1$

where r_1 and r_2 are the smallest positive roots of the following equations respectively:

$$(1 + (2\alpha - 1) r)(1 - Ar)(1 - Br) - (A - B) r (1 + r) = 0 \quad \ldots(31)$$

and

$$(1 + (2\alpha - 1) r)(A - B)(1 - r^2) + (1 + r)(1 - r^2)(A + B) + 2(1 + r)[(L_1 K_1)^{1/2} - (1 - ABr^2)] = 0. \quad \ldots(32)$$

PROOF : This is an immediate consequence of Theorem 6. The result is sharp.

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