Indian J. pure appl. Math., 13(4) : 471-475, April 1982

ON THE DEGREE OF APPROXIMATION TO A FUNCTION BELONGING TO WEIGHTED (L^p, $\psi_1(t)$) CLASS

KUTBUDDIN QURESHI

Department of Mathematics, Govt. P.G. College, Narsinghpur (M.P.)

(*Received* 16 *February* 1981; *after revision* 22 *July* 1981)

In this paper the author has determined the degree of approximation of certain functions belonging to the weighted (L^p, $\psi_1(t)$) class by Nörlund means of its conjugate series.

§ 1. Let f be periodic with period 2π and integrable in the lebesgue sense. Let its Fourier series be given by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \ldots(1.1)$$

The conjugate series of the Fourier series (1.1) is given by,

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx). \qquad \ldots(1.2)$$

Let $\{p_n\}$ be a nonnegative, non-increasing generating sequence for the (N, p_n) method such that

$$P_n \equiv P(n) = p_0 + p_1 + p_2 + \ldots + p_n \to \infty \text{ as } n \to \infty. \qquad \ldots(1.3)$$

We write $p(y) = P_{[y]}$ and $P(y) = p_{[y]}$

where $[y]$ as usual denotes the greatest integer less than y. We define the norm by

$$|| f ||_p = \{\int_0^{2\pi} | f(x) |^p dx\}^{1/p}, p \geqslant 1 \qquad \ldots(1.4)$$

and let the degree of approximation be given by (see Zygmund 1959)

$$E_n(f) = \min_{T_n} || f - T_n ||_p \qquad \ldots(1.5)$$

where $T_n(x)$ is some nth degree trigonometric polynomial.

Given a positive increasing function $\psi_1(t)$ and an integer $p > 1$, we find (Siddiqi 1967) that $f(x) \in \text{Lip}\,(\psi_1(t), p)$ if

$$\{\int_0^{2\pi} | f(x+t) - f(x) |^p dx\}^{1/p} = O(\psi_1(t))$$

and that $f(x) \in W(L^p, \psi_1(t))$

if

$$\{\int_0^{2\pi} | f(x+t) - f(x) |^p \sin^{\beta p} x \, dx\}^{1/p} = O(\psi_1(t)), (\beta \geqslant 0).$$

In case $\beta=0$, we notice that our newly defined class $W(L^p, \psi_1(t))$ coincides with the class Lip $(\psi_1(t),p)$.

We proved the following theorems (see Qureshi 1981a,b):

Theorem A—If the sequence $\{p_n\}$ satisfies the following conditions

$$n \mid p_n \mid < C \mid P_n \mid \quad ...(2.1)$$

$$\sum_{k=1}^{n} k \mid p_k - p_{k-1} \mid < C \mid P_n \mid \quad ...(2.2)$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function f with period 2π and belonging to the class of Lip α, $0<\alpha<1$ by Nörlund means of its conjugate series, is given by

$$\left|\tilde{f}(x) - \tilde{t}_n(x)\right| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}\right)$$

where $\tilde{t}_{n(x)}$ are the (N, p_n) means of the series (1.2).

Theorem B—If $f(x)$ is periodic and belongs to the class Lip (α, p) for $0<\alpha \leqslant 1$, and if the sequence $\{p_n\}$ is as defined in (1.3) with the other requirements there in and if

$$\left(\int_1^n \frac{(p(y))^q}{y^{q\alpha+2+\delta q-q}}\, dy\right)^{1/q} = O\left(\frac{P(n)}{n^{\alpha+(1/q)+\delta-1}}\right)$$

then $\| \tilde{t}_n - \tilde{f} \|_p = O\left(\frac{1}{n^{\alpha-(1/p)}}\right)$

where $\tilde{t}_n$ are the (N, p_n) means of the series (1.2) and $1/p+1/q=1$ such that $1 \leqslant p \leqslant \infty$.

Our object of this paper is to prove the following theorem:

Theorem C—If a 2π periodic function belongs to the class $W(L^p, \psi_1(t))$, then its degree of approximation by Nörlund means of its conjugate series, is given by

$$\| \tilde{f}(x) - \tilde{t}_n(x) \|_p = O(\psi_1(1/n) n^{\beta+(1/p)})$$

provided $\psi_1(t)$ satisfies the following conditions:

$$(1) \quad \left\{\int_0^{\pi/n} \left(\frac{t \mid \psi(t) \mid}{\psi_1(t)}\right)^p \sin^{\beta p} t \, dt\right\}^{1/p} = O(1/n)$$

$$(2) \quad \left\{\int_{\pi/n}^{\pi} \left(\frac{t^{-\delta} \mid \psi(t) \mid}{\psi_1(t)}\right)^p dt\right\}^{1/p} = O(n^{\delta})$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, conditions (1) and (2) hold uniformly in x and

$$\psi(t) = f(x+t) - f(x-t).$$

$$(3)\quad \left\{\int_0^{\pi/n}\left(\frac{\psi_1(t)}{t^{2+\beta}}\right)^q dt\right\}^{1/q} = O\left(\psi(1/n)n^{\beta+1+(1/p)}\right)$$

where $1/p + 1/q = 1$ such that $1 \leqslant p \leqslant \infty$.

The following lemmas are known:

Lemma A (Sahney and Goel 1973, Lemma 1)

If the sequence $\{p_n\}$ is nonnegative and non-increasing, then for $\alpha > 0$

$$\frac{1}{n^{\alpha}} \leqslant \frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}.$$

Lemma B McFadden 1942, Lemma 5-(11)—If $\{p_n\}$ is nonnegative and non-increasing, then, for $0 \leqslant a \leqslant b \leqslant \infty$; $0 \leqslant t \leqslant \pi$ and any n, we have

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| \leqslant P(1/t) \text{ for any } a.$$

§ 3. *Proof of the Theorem*—Since

$$\tilde{S}_k(x) - \tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(k+\frac{1}{2})t}{2\sin t/2}\,dt,$$

we have

$$\tilde{t}_n(x) - \tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos(k+\frac{1}{2})t}{2\sin t/2}\,dt$$

$$= -\frac{1}{\pi P_n}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right]\frac{\psi(t)}{t} \times \sum_{k=0}^{n}p_{n-k}\cos(k+\tfrac{1}{2})\,t\,dt + O(1)$$

$$= I_1 + I_2 + O(1), \text{ say.}$$

Applying Hölders inequality and the fact that $\psi(t) \in W(L^p, \psi_1(t))$,

we get

$$I_1 = -\frac{1}{\pi P_n}\int_0^{\pi/n}\frac{\psi(t)}{t}\sum_{k=0}^{n}p_{n-k}\cos(k+\tfrac{1}{2})t\,dt$$

$$\leqslant O\left(\frac{1}{P_n}\right)\left\{\int_0^{\pi/n}\left[\left(\frac{t\,|\psi(t)|}{\psi_1(t)}\right)\sin^{\beta}t\right]^p dt\right\}^{1/p}$$

$$\times\left\{\int_0^{\pi/n}\left[\frac{\psi_1(t)}{t^2}\left|\sum_{k=0}^{n}p_{n-k}\frac{\cos(k+\frac{1}{2})t}{\sin^{\beta}t}\right|\right]^q dt\right\}^{1/q}$$

(*equation continued on p.* 474)

$$=O\left(\frac{1}{P_n}\right)O\left(\frac{1}{n}\right)\left\{\int_0^{\pi/n}\left[\frac{\psi_1(t)}{t^2}\times\left|\sum_{k=0}^{n}p_{n-k}\frac{1}{\sin^{\beta}t}\right|\right]^q dt\right\}^{1/q}$$ (by Condition (1))

$$=O\left(\frac{1}{n}\right)O\left\{\int_0^{\pi/n}\left[\frac{\psi_1(t)}{t^{\beta+2}}\right]^q dt\right\}^{1/q}$$

$$=O\left(\frac{1}{n}\right)O\left(\psi_1(1/n)n^{\beta+1+1/p}\right)$$ (by Condition (3))

$$=O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+1/p}\right).$$

Also, similarly, as above

$$I_2\leqslant O\left(\frac{1}{P_n}\right)\left\{\int_{\pi/n}^{\pi}\left|\frac{t^{-\delta}\sin^{\beta}t\psi(t)}{\psi_1(t)}\right|^p dt\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_{\pi/n}^{\pi}\left[\left|\sum_{k=0}^{n}p_{n-k}\frac{\cos(k+\frac{1}{2})t\psi_1(t)}{\sin^{\beta}t\ t^{-\delta+1}}\right|\right]^q dt\right\}^{\frac{1}{q}}$$

$$=O\left(\frac{1}{P_n}\right)\left\{\int_{\pi/n}^{\pi}\left[t^{-\delta}\frac{|\psi(t)|}{\psi_1(t)}\right]^p dt\right\}^{\frac{1}{p}}\times O\left\{\int_{\pi/n}^{\pi}\left[\frac{\psi_1(t)P(1/t)}{t^{-\delta+1}\sin^{\beta}t}\right]^q dt\right\}^{\frac{1}{q}}$$ (by Lemma B)

$$=O\left(\frac{1}{P_n}\right)O(n^{\delta})\frac{1}{\sin^{\beta}(\pi/n)}O\left\{\int_1^{n}\left[\frac{\psi_1(1/y)P(y)}{y^{\delta-1}}\right]\frac{dy}{y^2}\right\}^{\frac{1}{q}}$$ (by Condition (2))

$$=O\left(\frac{1}{P_n}\right)O(n^{\delta})O\left(\frac{P(n)\psi_1(1/n)}{\sin^{\beta}(1/n)}\right)O\left\{\int_1^{n}\frac{1}{y^{\delta q-q+2}}dy\right\}^{\frac{1}{q}}$$

$$=O(n^{\delta})O\left(\frac{\psi_1(1/n)}{(1/n)^{\beta}}\right)O\left(n^{-\delta q+q-1}\right)^{1/q}=O(n^{\delta})O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta}\right)O(n^{-\delta+1-(1/q)})$$

$$=O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+1-(1/q)}\right)=O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+(1/p)}\right)$$

Hence

$$\left|\tilde{t}_n(x)-\tilde{f}(x)\right|=O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+1/p}\right)$$

Therefore $\left|\left|\tilde{f}(x)-\tilde{t}_n(x)\right|\right|_p=O\left[\left\{\int_0^{2\pi}\left(\psi_1(1/n)n^{\beta+(1/p)}\right)^p dx\right\}^{1/p}\right]$

(*equation continued on p.* 475)

$$=O\left[\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+(1/p)}\right)\left\{\left(\int_0^{2\pi}dx\right)^{1/p}\right\}\right]$$

$$=O\left[\psi_1\left(\frac{1}{n}\right)n^{\beta+(1/p)}\right].$$

This completes the proof of the theorem.

§ 4. *Corollary*—If $p\to\infty$, $\beta=0$, $\psi(t)=t^{\alpha}$; using Lemma A, we have Theorem A.

PROOF:

$$\|\tilde{f}(x)-\tilde{t}_n(x)\|_p=\{\int_0^{2\pi}|\tilde{f}(x)-\tilde{t}_n(x)|^p dx\}^{1/p}$$

$$O\left(\psi_1\left(\frac{1}{n}\right)n^{\beta+(1/p)}\right)=\left\{\int_0^{2\pi}|\tilde{f}(x)-\tilde{t}_n(x)|^p dx\right\}^{1/p}$$

$$O(1)=\left\{\int_0^{2\pi}|\tilde{f}(x)-\tilde{t}_n{}^p(x)|^p dx\right\}^{1/p} O\left(\frac{1}{\psi_1\left(\frac{1}{n}\right)n^{\beta+(1/p)}}\right)$$

Hence $|\tilde{f}(x)-\tilde{t}_n(x)|=O(\psi_1(\frac{1}{n})n^{\beta+(1/p)})$

for if not the right hand side will not be $O(1)$.

Now taking $p\to\infty$, $\beta=0$ and $\psi(t)=t^{\alpha}$, we have

$$|\tilde{f}(x)-\tilde{t}_n(x)|=O[(\tfrac{1}{n})^{\alpha}]$$

$$=O\left[\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{\alpha+1}}\right] \qquad \text{(by Lemma A).}$$

Remark:—Taking $\beta=0$ and $\psi(t)=t^{\alpha}$, it is to be noted that Theorem C is equivalent to Theorem B.

ACKNOWLEDGEMENT

The author is very much thankful to Dr P.L. Sharma, Professor and Head, department of Mathematics, University of Saugar, for his valuable guidance in the preparation of the paper. He is also thankful to the referee for his kind suggestions for the improvement of the paper.

REFERENCES

McFadden, L. (1942). Absolute Nörlund summability. *Duke Math. J.*, 9, 168-207.

Qureshi, K. (1981a). On the degree of approximation of functions belonging to the Lipschitz class by means of a conjugate series. *Indian J. pure appl. Math.*, 9, 1120-23.

———(1981 b). On the degree of approximation of functions belonging to the class Lip (α,p) by means of a conjugate series. *Indian J. pure appl. Math.*, 4, 466-70.

Sahney, B.N., and Goel, D. S. (1973). On the degree of approximation of continuous functions. *Ranchi Univ. Math. J.*, 4, 50-53.

Siddiqi, A. H. (1967). Ph.D. thesis, Aligarh Muslim University, Aligarh.

Zygmund, A. (1959). Trigonometric Series, Vol. 1, 2nd ed. Cambridge University Press, Cambridge.