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INVERSE AND SATURATION THEOREMS FOR DERIVATIVES OF EXPONENTIAL TYPE OPERATORS

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The present paper is a study of the inverse and saturation theorems in the simultaneous approximation by certain linear combinations of exponential type operators. The inverse theorem has been obtained through the approach of Peetre's K-functionals, while for the saturation theorem, the operators dual to those of the exponential type have been used.

1. INTRODUCTION

Let $-\infty \leqslant A < B \leqslant \infty$ and Λ denote an unbounded subset of $\mathbb{R}^+ := (0, \infty)$. Let $\{S_\lambda, \lambda \in \Lambda\}$ be a family of operators mapping the space $C(A, B)$ of bounded continuous functions on (A, B) into $C^\infty(A, B)$ and having the form

$$S_\lambda(f, t) = \int_A^B W(\lambda, t, u) f(u)\, du \qquad \ldots(1)$$

where $W(\lambda, t, u) \geqslant 0$, $S_\lambda(1, t) = 1, t \in (A, B)$

$$\frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{p(t)} W(\lambda, t, u)(u-t), \quad u, t \in (A, B) \qquad \ldots(2)$$

where $p(t)$ is a polynomial of degree $\leqslant 2$ and is positive on (A, B) and that for all $k \in \mathbb{N} := \{1, 2, \ldots\}$

$$\frac{d^k}{dt^k} S_\lambda(f, t) = \int_A^B \frac{\partial^k}{\partial t^k} [W(\lambda, t, u)] f(u)\, du, \; t \in (A, B). \qquad \ldots(3)$$

The operators S_λ were introduced by May (1976) and are called exponential type operators. The integral in (1) is in the Lebesgue-Stieltjes sense and $W(\lambda, t, u)$ is a distributional kernel.

If, in addition, $W(\lambda, t, u)$ regarded as a function in t and u is measurable on $(A, B) \times (A, B)$,

$$\int_A^B W(\lambda, t, u)\, dt = a(\lambda), \; u \in (A, B) \qquad (4)$$

where $a(\lambda)$ is a rational function of λ with $a(\lambda) \to 1$ as $\lambda \to \infty$ and for each fixed $u \in (A, B)$, $m \in \mathbb{N}^0 := \{0, 1, 2, \ldots\}$ and all λ sufficiently large,

$$t^m p(t) W(\lambda, t, u) \to 0 \quad \text{as } t \to A, B \qquad \ldots(5)$$

we say that the operators S_λ are regular.

In this paper we study the approximation of derivatives of f by the derivatives of the linear combinations $S_\lambda(f, k, t)$ defined by

$$S_\lambda(f, k, t) := \begin{vmatrix} 1 & d_0^{-1} & \cdots & d_0^{-k} \\ 1 & d_1^{-1} & \cdots & d_1^{-k} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & d_k^{-1} & \cdots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} S_{d_0\lambda}(f, t) & d_0^{-1} & \cdots & d_0^{-k} \\ S_{d_1\lambda}(f, t) & d_1^{-1} & \cdots & d_1^{-k} \\ \cdots & \cdots & \cdots & \cdots \\ S_{d_k\lambda}(f, t) & d_k^{-1} & \cdots & d_k^{-k} \end{vmatrix} \quad \ldots (6)$$

where d_i, $i = 0, 1, \ldots, k$ are distinct positive real numbers such that $d_i \lambda \in \Lambda$, $i = 0, 1, \ldots, k$ for $\lambda \in \Lambda^0$, an unbounded subset of $\mathbb{R}^+$. Combinations of this type were first considered by Rathore (1973) and it is easily seen that they can be alternately written in the following form

$$S_\lambda(f, k, t) = \sum_{j=0}^{k} \prod_{\substack{i=0 \\ i \neq j}}^{k} \left(\frac{d_j}{d_j - d_i} \right) S_{d_j\lambda}(f, t). \quad \ldots (7)$$

For $k = 0$, we define $S_\lambda(f, k, t) = S_{d_0\lambda}(f, t)$.

May (1976) obtained inverse and saturation (in the regular case) theorems for $S_\lambda(f, k, t)$ in the ordinary approximation. We shall extend his results to the simultaneous approximation by proving the following inverse and saturation results, further extensions of which have been obtained by Agrawal (1979).

Let $m \in N$, $k \in \mathbb{N}^0$, a_i, b_i, $(i = 1, 2, 3)$ be such that $A < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < B$ and $\{\lambda_n \in \Lambda^0\}_{n \in \mathbb{N}}$ denote an unbounded sequence. Also, let $\langle a, b \rangle$ denote an open interval containing the closed interval $[a, b]$ and $D \equiv d/dt$, the differentiation symbol.

Theorem 1 (*Inverse theorem*)—Let $0 < \alpha < 2$ and $\lambda_{n+1}/\lambda_n \leqslant C$, $n \in \mathbb{N}$. Then, for $f \in C(A, B)$, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) where

(i) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\sup_{t \in [a_1, b_1]} | S^{(m)}_{\lambda_n}(f, k, t) - f^{(m)}(t) | = O(\lambda_n^{-\alpha(k+1)/2});$$

(ii) $f^{(m)} \in \text{Liz}(\alpha, k+1; a_2, b_2)$;

(iii) (a) If $p < \alpha(k+1) < p + 1$, $p = 0, 1, 2, \ldots, 2k + 1$, then $f^{(p+m)}$ exists and $\in \text{Lip}(\alpha(k+1) - p; a_2, b_2)$,

(b) If $\alpha(k + 1) = p + 1$, $p = 0, 1, \ldots, 2k$, then $f^{(p+m)}$ exists and $\in \text{Lip}^*(1; a_2, b_2)$:

(iv) $\| S_\lambda^{(m)}(f, k, t) - f^{(m)}(t) \|_{C[a_3, b_3]} = O(\lambda^{-\alpha(k+1)/2})$.

Here the classes $\text{Liz}(\alpha, k; a, b)$, $\text{Lip}(\alpha; a, b)$ and $\text{Lip}^*(1; a, b)$ are defined as

$\text{Liz}(\alpha, k; a, b) = \{f \in C[a, b] : \omega_{2k}(f; \delta) = O(\delta^{\alpha k}), \delta \to 0\}$,

$\text{Lip}(\alpha; a, b) = \{f \in C[a, b] : \omega_1(f; \delta) = 0(\delta^\alpha), \delta \to 0\}$

and

$\text{Lip}^*(1; a, b) = \{f \in C[a, b] : \omega_2(f; \delta) = O(\delta), \delta \to 0\}$,

where for $m = 1, 2, \ldots$, the modulus of continuity $\omega_m(f; \delta) = \omega_m(f, a, b; \delta)$ is given by

$$\omega_m(f; \delta) = \sup_{|h| \leqslant \delta} \left\{ \left| \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh) \right| : x, x + mh \in [a, b] \right\}.$$

Theorem 2 (*Saturation theorem*)—If S_λ are regular and $f \in C(A, B)$, then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) where

(i) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1, b_1]} | S_{\lambda_n}^{(m)}(f, k, t) - f^{(m)}(t) | = O(1);$$

(ii) $f^{(2k+m+1)} \in A.\ C.\ [a_2, b_2]$ and $f^{(2k+m+2)} \in L_\infty [a_2, b_2]$;

(iii) $\lambda^{k+1} \| S_\lambda^{(m)}(f, k, t) - f^{(m)}(t) \|_{C[a_3, b_3]} = o(1)$;

(iv) $f^{(m)}$ exists on $[a_1, b_1]$ and

$$\lambda_n^{k+1} \sup_{t \in [a_1, b_1]} | S_{\lambda_n}^{(m)}(f, k, t) - f^{(m)}(t) | = o(1);$$

(v) $f \in C^{2k+m+2}[a_2, b_2]$ and $\sum_{i=m}^{2k+m+2} Q(i, k, m, t) f^{(i)}(t) = 0$,

$t \in [a_2, b_2]$ where $Q(i,k,m,t)$ are the polynomials occurring in (14);

(vi) $\lambda^{k+1} \| S_\lambda^{(m)}(f, k, t) - f^{(m)}(t) \|_{C[a_3, b_3]} = o(1)$.

2. Auxiliary Results

In case the operators S_λ are regular we define the associated dual operators

$$S_\lambda^* \text{ by } \quad S_\lambda^*(f(t), u) = \int_A^B W(\lambda, t, u) f(t)\, dt \qquad \ldots(8)$$

the domain of S_λ^* being $C(A, B)$. If $\langle f, g \rangle$ denotes the inner product $\int_A^B f(t)\, g(t)\, dt$, by Fubini's theorem it is easily seen that $\langle S_\lambda f, g \rangle = \langle f, S_\lambda^* g \rangle$. Also, if the normalized kth moment $\sigma_{\lambda,k}^*(u)$ of S_λ^* is defined by

$$\sigma_{\lambda,k}^*(u) = [a(\lambda)]^{-1} \mu_{\lambda,k}^*(u), \; k \in \mathbb{N}^0, \qquad \ldots(9)$$

where the kth moment $\mu_{\lambda,k}^*(u) = S_\lambda^*((t-u)^k, u)$, we have the following.

Lemma 1—For each $k \in \mathbb{N}^r$, there holds

$$\{\lambda - (k+2)\alpha\} \sigma_{\lambda,k+1}^*(u) = 2(k+1)(\alpha u + \beta) \sigma_{\lambda,k}^*(u) + kp(u) \sigma_{\lambda,k-1}^*(u), \qquad \ldots(10)$$

for all λ sufficiently large, where $p(t) = \alpha t^2 + 2\beta t + \gamma$ is as in (2) and $\sigma_{\lambda,-1}^*(u) \equiv 0$.

Proof : By definition, for all λ sufficiently large we have

$$\sigma_{\lambda,k+1}^*(u) = [a(\lambda)]^{-1} \int_A^B W(\lambda, t, u)(t-u)^{k+1} dt$$

$$= -[a(\lambda)]^{-1} \int_A^B \left(\frac{p(t)}{\lambda} \frac{\partial}{\partial t} W(\lambda, t, u) \right) (t-u)^k du$$

(*equation continued on p.* 479)

$$= [a(\lambda)]^{-1} \int_A^B \left(\frac{p(t)(t-u)^k}{\lambda} \right)'_t W(\lambda, t, u)\, du$$

using integration by parts, (2) and (5). Hence

$$\sigma^*_{\lambda, k+1}(u) = [\lambda\, a(\lambda)]^{-1} \int_A^B [\alpha (t-u)^{k+2} + 2(\alpha u + \beta)(t-u)^{k+1}$$

$$+ p(u)\,(t-u)^k]'_t\; W(\lambda, t, u)\, dt$$

$$= [\lambda\, a(\lambda)]^{-1} \int_A^B [\alpha (k+2)(t-u)^{k+1} + 2(k+1)(\alpha u + \beta)(t-u)^k$$

$$+ kp(u)(t-u)^{k-1}]\; W(\lambda, t, u)\, dt,$$

where for $k = 0$ the last term within the square brackets is absent.

From this (10) is immediate.

Using Lemma 1, by induction it can be easily shown that for all λ sufficiently large, $\sigma^*_{\lambda,k}(u)$ is a polynomial in u of degree $\leqslant k$ and moreover that $\sigma^*_{\lambda,k}(u) = O(\lambda^{-[k+(1/2)]})$.

An asymptotic formula for S^*_λ which we shall require in the proof of the saturation theorem is given by

Theorem 3—Let $f \in C(A, B)$ and $f^{(2k+2)}$ exists at a point $u \in (A, B)$. Then,

$$S^*_\lambda(f, u) - f(u) = (a(\lambda) - 1) f(u) + \sum_{j=1}^{2k+2} \frac{f^{(j)}(u)}{j!}\, \mu^*_{\lambda, j}(u) + o(\lambda^{-(k+1)}). \qquad \text{...(11)}$$

Moreover, if $f^{(2k+2)}$ exists and is continuous on $\langle a, b \rangle$ then (11) holds uniformly in $u \in [a, b]$.

The theorem easily follows from Rathore (1974, p. 115, Th. 3).

In the rest of this section we give some direct results on the simultaneous approximation property of $S_\lambda(f, t)$ which are also needed in the proofs of Theorems 1 and 2.

Lemma 2 (*Lorentz type*)—For each $k \in \mathbb{N}^0$, there exist polynomials $q_{ij}^{[k]}(t)$ in t not depending on u or λ such that

$$\frac{\partial^k}{\partial t^k} W(\lambda, t, u) = Q_k(t, u, \lambda)\, W(\lambda, t, u) \qquad \text{...(12)}$$

where $Q_k(t, u, \lambda) = \displaystyle\sum_{\substack{2i+j \leqslant k \\ i,j \geqslant 0}} \lambda^{i+j} (u-t)^j \frac{q_{ij}^{[k]}(t)}{(p(t))^k}.$

The proof of the lemma makes use of (2) and easily follows along the lines of the proof of Lemma of Lorentz (1953, p. 26).

Making use of Lemma 2 and following the proof of [Th. 1.8.1 of Lorentz (1953, p. 26)] one easily proves the following basic convergence result.

Theorem 4—If $f \in C(A, B)$ and $f^{(m)}$ exists at a point $t \in (A, B)$, then

$$\lim_{\lambda \to \infty} S_\lambda^{(m)}(f, k, t) = f^{(m)}(t). \quad \ldots(13)$$

Further, if $f^{(m)}$ exists and is continuous on $\langle a, b \rangle$ then (13) holds uniformly in $t \in [a, b]$.

Lemma 3—If $\mu_{\lambda,k}(t) = S_\lambda((u-t)^k, t)$, then $\mu_{\lambda,k}(t)$ is a polynomial in t of degree $\leqslant k$ and of degree $\leqslant k$ in λ^{-1} with $\mu_{\lambda,k}(t) = O(\lambda^{-[k+1/2]})$. The coefficient of λ^{-k} in $\mu_{\lambda,2k}(t)$ is $(2k-1)$!—! $p^k(t)$, a!! denoting the semifactorial of a. Also, the coefficient of λ^{-k} in $\mu_{\lambda,2k+1}(t)$ is $(2k+1)!!\, k/3\, p^k(t)\, p'(t)$.

The proof of Lemma 3 follows from May (1976, Prop. 3.2).

Theorem 5—If $m, k \in \mathbb{N}^0$, $f \in C(A, B)$ and $f^{(2k+m+2)}$ exists at a point $t \in (A, B)$, then

$$\lim_{\lambda \to \infty} \lambda^{k+1}[S_\lambda^{(m)}(f, k, t) - f^{(m)}(t)] = \sum_{j=m}^{2k+m+2} Q(j, k, m, t) f^{(j)}(t) \quad \ldots(14)$$

and $$\lim_{\lambda \to \infty} \lambda^{k+1}[S_\lambda^{(m)}(f, k+1, t) - f^{(m)}(t)] = 0 \quad \ldots(15)$$

where $Q(j, k, m, t)$ are certain polynomials in t with $Q(2k+m+2, k, m, t)$

$$= (-1)^k p(t)^{k+1}/(2k+2)!! \prod_{i=0}^{k} d_i.$$

Also, if $f^{(2k+m+2)}$ exists and is continuous on $\langle a, b \rangle$, then (14) and (15) hold uniformly in $t \in [a, b]$.

For $m = 0$, theorem reduces to Prop. 3.6 of May (1976). For $m \geqslant 1$ the proof makes use of Lemma 2 and can be given essentially along the lines of the proofs of Theorem 6 of Rathore (1976) or Theorem 3.1 of Rathore (1978).

Theorem 6—Let $m \leqslant q \leqslant 2k + m + 2$, $f \in C(A, B)$ and $f^{(q)}$ exist and be continuous on $\langle a, b \rangle$. Then,

$$\| S_\lambda^{(m)}(f, k, t) - f^{(m)}(t) \|_{C[a,b]} \leqslant \max \{C \lambda^{-(q-m)/2} \omega(f^{(q)}; \lambda^{-1/2}), C' \lambda^{-(k+1)}\}, \quad \ldots(16)$$

where $C = C(k, m)$, $C' = C'(k, m, f)$ and $\omega(f^{(q)}; \delta)$ denotes the modulus of continuity of $f^{(q)}$ on $\langle a, b \rangle$.

PROOF: For $u \in (A, B)$ and $t \in [a, b]$, we can write

$$f(u) = \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(q)}(\xi) - f^{(q)}(t)}{q!}(u-t)^q x(u) + h(u, t)(1 - x(u)), \quad \ldots(17)$$

where ξ lies between u and t and $x(u)$ is the characteristic function of $\langle a, b \rangle$. The function $h(u, t)$ for $t \in [a, b]$ is bounded by $M(u-t)^q$ for some constant M. Operating on this equality by $S_\lambda^{(m)}(., k, t)$ and breaking the right-hand side into three parts I_1, I_2 and I_3, say, corresponding to the three terms on the right-hand side of (17), by Theorem 5 it follows that

$$I_1 = \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} t^{i-j} S_\lambda^{(m)}(u^j, k, t)$$

$$= \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} (\ \ 1)^{i-j}\binom{i}{j} t^{i-j} [D^m t^j + O(\lambda^{-(k+1)})]$$

$$= \sum_{j=0}^{q} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j} m! t^{i-m} + O(\lambda^{-(k+1)})$$

$$= f^{(m)}(t) + O(\lambda^{-(k+1)})$$

using the identities $\sum_{l=0}^{i} (-1)^l \binom{i}{l}\binom{l}{p} = \begin{cases} 0 & , i > p \\ (-1)^p, & i = p \end{cases}$.

Also, by Lemmas 2 and 3 and the Cauchy Schwarz inequality it is easy to see that

$$I_3 = o(\lambda^{-(k+1)}).$$

Finally, again by Lemmas 2-3 and Cauchy Schwarz inequality

$$|I_2| \leqslant \sum_{j=0}^{k} |C(j, k)| \sum_{\substack{2r+s\leqslant m \\ r,s\geqslant 0}} (d_j\lambda)^{r+s} \frac{|q_{rs}^{[m]}(t)|}{(p(t))^m} \int_A^B W(d_j\lambda, t, u)$$

$$\times \frac{|f^{(q)}(\xi) - f^{(q)}(t)|}{q!} |u-t|^{q+s} x(u)\, du$$

$$\leqslant \frac{(f^{(q)}; \lambda^{-1/2})}{q!} \sum_{j=0}^{k} |C(j, k)| \sum_{\substack{2r+s\leqslant m \\ r,s\geqslant 0}} (d_j\lambda)^{r+s} \frac{|q_{rs}^{[m]}(t)|}{(p(t))^m}$$

$$\times \int_A^B W(d_j\lambda, t, u) |u-t|^s (|u-t|^q + \lambda^{1/2}|u-t|^{q-1})\, du$$

$$= \omega(f^{(q)}; \lambda^{-1/2})\, O(\lambda^{-(q-m)/2}),$$

uniformly in $t \in [a, b]$.

Combining these estimates of $I_1 - I_3$ we obtain the required result.

3 Proof of the Inverse Theorem

Let $[a, b]$ be a fixed subinterval of (A, B) and let $[a', b'] \subset (a, b)$. Also, let $G^{(m)} := \{g : g \in C_0^{2k+m+2}[a', b']\}$. For $f \in C_0^m [a', b']$ let a Peetre's K-functional be defined by

$$K_m(\xi, f) = \inf_{g\in G^{(m)}} \{\|f^{(m)} - g^{(m)}\| + \xi(\|g\| + \|g^{(2k+m+2)}\|)\} \quad \ldots(18)$$

where $0 < \xi \leqslant 1$ and the norms are the sup-norms on $[a', b']$.

For $0 < \alpha < 2$, we define $C_0^m\ (\alpha, k+1; a', b')$ as the class of all $f \in C_0^m\ [a', b']$ such that the functional

$$\| f \|_{\alpha,m} := \sup_{0<\xi\leqslant 1} \xi^{-\alpha/2} K_m(\xi, f) < M \qquad \ldots(19)$$

for some constant $M > 0$.

Lemma 4—Let $a < a' < a'' < b'' < b' < b$. If $f \in C_0^m\ [a'', b'']$ and $\| S_{\lambda_n}^{(m)} (f, k, t) - f^{(m)}(t) \|_{C[a',b']} \leqslant M \lambda_n^{-\alpha(k+1)/2}$, then

$$K_m(\xi, f) \leqslant M_0 [\lambda^{-\alpha(k+1)/2} + \lambda^{k+1} \xi K_m(\lambda^{-(k+1)}, f)]. \qquad \ldots(20)$$

Consequently (cf. Berens and Lorentz 1972, p. 696 or Becker and Nessel 1978, p. 100), $K_m(\xi, f) \leqslant M' \xi^{\alpha/2}$ for some constant M' i.e. $f \in C_0^m\ (\alpha, k+1; a', b')$.

PROOF: To establish (20), it is enough to show that

$$K_m(\xi, f) \leqslant M_0 [\lambda_n^{-\alpha(k+1)/2} + \lambda_n^{k+1} \xi K_m(\lambda^{-(k+1)}, f)] \qquad \ldots(21)$$

for all n sufficiently large. This is because for each λ sufficiently large we can choose λ_{n-1} and λ_n such that $\lambda_{n-1} < \lambda \leqslant \lambda_n$.

Now, since $[a'', b''] \subset (a', b')$, we can choose a $\delta > 0$ such that $(a''-2\delta, b''+2\delta) \subset [a',b']$. Let g be an infinitely differentiable function such that $g(t) = 1$ on $[a''-2\delta, b''+2\delta]$ and $g(t) = 0$ on $(A, B)\setminus(a', b')$. Defining h_n to be equal to $g(t) S_{\lambda_n}(f,k,t)$, then $h_n \in G^{(m)}$. Since supp $f \subset [a'', b'']$ by Theorem 5 (applied on $[a, b]\setminus[a''-2\delta, b''2\delta]$) one has

$$\| h_n^{(i)}(t) - S_{\lambda_n}^{(i)}(f, k, t) \|_{C[a,b]} \leqslant M_1 \lambda_n^{-(k+1)}, \quad i=m, 2k+m+2, \qquad \ldots(22)$$

for all n sufficiently large.

Therefore, for some constant M_1

$$K_m(\xi,f) \leqslant 2M_1\lambda_n^{-(k+1)} + \| f^{(m)}(t) - S_{\lambda_n}^{(m)}(f, k, t) \|_{C[a',b']} + \xi \left[\| h_n(t) \|_{C[a',b']} + \| S_{\lambda_n}^{(2k+m+2)}(f, k, t) \|_{C[a',b']} \right].$$

Hence it is enough to show that there exists an M_2, such that, for each $g \in G^{(m)}$

$$\| S_\lambda^{(2k+m+2)}(f, k,t) \|_{C[a',b']} \leqslant M_2\lambda^{k+1}[\| f^{(m)} - g^{(m)} \|_{C[a',b']} + \lambda^{-(k+1)} \| g^{(k+m+2)} \|_{C[a',b']}] \qquad \ldots(23)$$

We can write

$$\| S_\lambda^{2k+m+2}(f, k, t) \|_{C[a',b']}$$

$$\leqslant \sum_{j=0}^{k} | C(j,k) | \, \| \int_A^B W^{(2k+m+2)}(d_j\lambda, t,u)\,(f-g)(u)du \|_{C[a',b']}$$

$$+ \sum_{j=0}^{k} | C(j,k) | \, \| \int_A^B W^{(2k+m+2)}(d_j\lambda,t,u)g(u)\,du \|_{C[a',b']}$$

$$= I_1 + I_2, \text{ say} \qquad \ldots(24)$$

where $C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}$.

To estimate I_1, by a Taylor expansion of $(f-g)$ we have

$$(f-g)(u) = \sum_{i=0}^{m-1} \frac{(f-g)^{(i)}(t)}{i!}(u-t)^i + \frac{(f-g)^{(m)}(\xi)}{m!}(u-t)^m,$$

where ξ lies between u and t. Then, by Lemmas 2 and 3 and Cauchy-Schwarz inequality it is easily seen that

$$I_1 \leq M_3 \lambda^{k+1} \| f^{(m)} - g^{(m)} \|_{C[a',b']} \quad (\operatorname{supp} f \cup \operatorname{supp} g \subset [a',b']), \qquad \ldots(25)$$

where M_3 is independent of g and f.

Similarly, from the Taylor expansion

$$g(u) = \sum_{i=0}^{2k+m+1} \frac{g^{(i)}(t)}{i!}(u-t)^i + \frac{g^{(2k+m+2)}(\eta)}{(2k+m+2)!}(u-t)^{2k+m+2},$$

where η lies between u and t, there follows

$$I_2 \leq M_4 \| g^{(2k+m+2)} \|_{C[a',b']} \quad . \qquad \ldots(26)$$

Combining these estimates we obtain (21).

Lemma 5—Let $a<a'<a''<b''<b'<b$. If $f \in C_0^m[a'',b'']$ then $f \in C_0^m(\alpha, k+1, a', b')$ iff $f^{(m)} \in \text{Liz}(\alpha, k+1; a',b')$.

PROOF: If $f \in C_0^m(\alpha, k+1; a',b')$. Then it is easy to see that $f^{(m)} \in \text{Liz}(\alpha, k+1; a',b')$. Conversely, let $f^{(m)} \in \text{Liz}(\alpha, k+1; a',b')$.

For the case, when m is odd, we define $g_0 \in G^{(m)}$ by

$$g_0(t) = 1 \Big/ \binom{2k+m+2}{k+[m/2]+1} \eta^{2k+m+2} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} \int_{-\eta/2}^{\eta/2} [(-1)^{k+[m/2]}$$

$$\times \Delta^{2k+m+2}_{\sum_{\nu=1}^{2k+m+2} u_\nu} f(t-\tfrac{1}{2}) + \binom{2k+m+2}{k+[m/2]+1} f(t)] \, du_1 du_2 \ldots du_{2k+m+2}, \qquad \ldots(27)$$

where $(k+[m/2]+1)^2 \eta < \min(a''-a', b'-b'')$ and Δ_h^r is the rth symmetric difference operator, and for the case when m is even, we define $g_0 \in G^{(m)}$ in a similar manner, the only difference being that we replace $f(t-\frac{1}{2})$ in (27) by $f(t)$. For both these cases, it is easily seen that

$$\left\| g_0^{(2k+m+2)} \right\|_{C[a',b']} \leq M_1 \eta^{-(2k+2)+\alpha(k+1)} \qquad \ldots(28)$$

$$\left\| g_0^{(m)} - f^{(m)} \right\|_{C[a',b']} \leq M_2 \eta^{\alpha(k+1)}. \qquad \ldots(29)$$

From these estimates it is clear that

$$K_m(\eta^{2k+2}, f) \leq M_2 \eta^{\alpha(k+1)}.$$

Hence $f \in C_0^m(\alpha, k+1;\ a', b')$ which completes the proof.

Proof of Theorem 1—It is enough to prove the implications '(ii) $\Rightarrow$ (iv)' and '(i) $\Rightarrow$ (ii)', the equivalence of (ii) and (iii) being well known [May 1976, p. 1230]. First let us assume (ii). Let $a_2 < a^* < a' < a'' < a_3 < b_3 < b'' < b' < b^* < b_2$ and $g \in C_0^\infty[a', b']$ be such that $g(t) = 1$ on $[a'', b'']$. Then, since $f^{(m)} \in$ Liz $(\alpha, k+1;\ a_2, b_2)$, also $(f g)^{(m)} \in$ Liz $(\alpha, k+1;\ a_2, b_2)$ and supp $fg \subset [a', b']$. Hence, by Lemma 4, $fg \in C_0^m(\alpha, k+1;\ a_2, b_2)$.

Using Theorem 5, we have

$$\begin{aligned} &\| S_\lambda^{(m)}(f,k,t) - f^{(m)}(t) \|_{C[a_3,b_3]} \\ &\leqslant \| S_\lambda^{(m)}(f - fg, k, t) \|_{C[a_3,b_3]} + \| S_\lambda^{(m)}(fg,k,t) - (fg)^{(m)}(t) \|_{C[a_3,b_3]} \\ &\leqslant \| S_\lambda^{(m)}(fg,k,t) - (fg)^{(m)}(t) \|_{C[a_3,b_3]} + O(\lambda^{-(k+1)}). \end{aligned} \qquad \ldots(30)$$

Now, for any $g_0 \in G^{(m)}$

$$\begin{aligned} &\| S_\lambda^{(m)}(fg,k,t) - (fg)^{(m)}(t) \|_{C[a^*,b^*]} \\ &\leqslant \| S_\lambda^{(m)}(fg - g_0, k, t) \|_{C[a^*,b^*]} + \| S_\lambda^{(m)}(g_0,k,t) - (fgt)^{(m)}(t) \|_{C[a^*,b^*]} \\ &= I_1 + I_2, \text{ say.} \end{aligned} \qquad \ldots(31)$$

Since supp $(fg - g_0) \subset [a', b']$, it is clear that

$$I_1 \leqslant M_1 \| fg - g_0 \|_{C[a',b']}. \qquad \ldots(32)$$

To estimate I_2, by Theorem 5 we have

$$\begin{aligned} I_2 &\leqslant \| g_0^{(m)} - (fg)^{(m)} \|_{C[a',b']} + M_2 \lambda^{-(k+1)} \sum_{j=m}^{2k+m+2} \| g_0^{(j)} \|_{C[a',b']} \\ &\leqslant \| g_0^{(m)} - (fg)^{(m)} \|_{C[a',b']} + M_3 \lambda^{-(k+1)} \Big(\| g_0 \|_{C[a',b']} \\ &\qquad + \| g_{(0)}^{2k+m+2} \|_{C[a',b']} \Big) \end{aligned} \qquad \ldots(33)$$

where M_3 is a certain constant.

Consequently, since $fg \equiv f$ on $[a'', b'']$ and $[a'', b''] \subset (a_3, b_3)$ it follows from (30) (33) that

$$\begin{aligned} \| S_\lambda^{(m)}(f,k,t) \quad f^{(m)}(t) \|_{C[a_3,b_3]} &\leqslant M_4 (K_m(\lambda^{-(k+1)}, f) + o(\lambda^{-(k+1)}) \\ &\leqslant M \lambda^{-\alpha(k+1)/2} \end{aligned}$$

proving the implication (ii) $\Rightarrow$ (iv). Next, let us assume (i). Notice that since $S_{\lambda_n} f$ are infinitely differentiable functions, by (i), $f^{(m)} \in C[a_1, b_1]$ and hence

$$\sup_{t \in [a_1, b_1]} | S_{\lambda_n}^{(m)}(f,k,t) - f^{(m)}(t) | = \| S_{\lambda_n}^{(m)}(f,k,t) - f^{(m)}(t) \|_{C[a_1,b_1]}$$

Put $\tau = \alpha(k+1)$. We shall prove the implication (i) $\Rightarrow$ (ii) by an induction on τ. First we consider the case when $0 < \tau \leqslant 1$. Let $g \in C_0^\infty[a'', b'']$ be such that $g(t) = 1$ on $[a_2, b_2]$ where $a_1 < a' < a'' < a_2 < b_2 < b'' < b' < b_1$. For $t \in [a', b']$, we have

$$S\lambda_n^{(m)}(fg,k,t)-(fg)^{(m)}(t)$$

$$= D^m[S\lambda_n(fg(u)-fg(t),\ k,\ t)]$$

$$= D^m[S\lambda_n(f(u)(g(u)-g(t)),\ k,\ t)] + D^m\,[S\lambda_n(g(t)\,f(u)-f(t)),\ k,\ t)]$$

$$= I_1+I_2, \text{ say.}$$

To estimate I_1, by Leibniz theorem ...(34)

$$I_1 = -\sum_{i=0}^{m-1}\binom{m}{i}g^{(m-i)}(t)\ S\lambda_n^{(i)}(f,\ k,\ t)$$

$$+\sum_{j=0}^{k}C(j,k)\int_A^B W^{(m)}(d_j\lambda_n,\ \ tu)\,f(u)\,(g(u)-g(t))du$$

$$= I_3+I_4, \text{ say.} \qquad ...(35)$$

Then, by Theorem 6 we get

$$I_3 = -\sum_{i=0}^{m-1}\binom{m}{i}g^{(m-i)}(t)f^{(i)}(t) + O(\lambda_n^{-\tau/2}) \qquad ...(36)$$

uniformly in $t\in[a',b']$.

Next substituting the Taylor expansions

$$f(u) = \sum_{i=0}^{m}\frac{f^{(i)}(t)}{i!}(u-t)^i+o(u-t)^m$$

$$\text{and}\quad g(u) = \sum_{i=0}^{m+1}\frac{g^{(i)}(t)}{i!}(u-t)^i+o(u-t)^{m+1}$$

in I_4 and using Theorem 5, Schwarz inequality and Lemmas 2 and 3 we get

$$I_4 = \sum_{i=1}^{m}\binom{m}{i}g^{(i)}(t)\,f^{(m-i)}(t)+O(\ \lambda_n^{-\tau/2}), \qquad ...(37)$$

uniformly in $t\in[a',b']$.

Lastly, by Leibniz theorem, Theorem 6 and the hypothesis that (i) holds we obtain

$$I_2 = O\left(\ \lambda_n^{-\tau/2}\right) \qquad ...(38)$$

uniformly in $t\in[a',b']$.

Combining (34-38) it follows that

$$||\ S_{\lambda_n}^{(m)}(\,fg,\ k,\ t)-(\,fg)^{(m)}(t)\ ||_{[Ca',b']} = O\ (\lambda_n^{-\tau/2}).$$

Thus, by Lemmas 4-5 we have $(fg)^{(m)} \in$ Liz $(\alpha, k+1; a', b')$. Since $g(t)=1$ on $[a_2, b_2]$ and $[a_2,b_2]\subset(a',b')$ hence $f^{(m)} \in$ Liz $(\alpha, k+1; a_2,b_2)$ proving the implication (i)$\Rightarrow$(ii) when $0<\tau\leqslant 1$.

To prove '(i)$\Rightarrow$(ii)' for the general case $0<\tau<2k+2$, it is sufficient to assume it for $\tau\in(p-1,p)$ and prove it for $\tau\in[p,p+1)$, $p=1,2,\ldots,2k+1$. Let a_1^*, b_1^*, a_2^* and b_2^* be such that $a_1<a_1^*<a_2^*<a_2<b_2<b_2^*<b_1^*<b_1$. Also, let $g\in C_0^\infty(a_2^*, b_2^*)$ be such that $g(t)=1$ on $[a_2, b_2]$. Now, assuming that $\tau\in[p,p+1)$ and (i) holds, in view of the assumption (i) $\Rightarrow$ (ii) for $\tau\in(p-1,p)$ and the equivalence of (ii) and (iii) it follows that $f^{(m+p-1)}$ exists and $\in$ Lip $(1-\delta; a_1^*, b_1^*)$ for any $\delta>0$. Then,

$$\begin{aligned}
&\| S\lambda_n^{(m)}(fg,k,t)-(fg)^{(m)}(t)\|_{C[a_2^*,b_2^*]}\\
&\leqslant \| D^m[S\lambda_n(g(t)(f(u)-f(t)),k,t)]\|_{C[a_2^*,b_2^*]}\\
&+\| D^m[S_n^\lambda(f(u)(g(u)-g(t)),k,t)\|_{C[a_2^*,b_2^*]}\\
&=I_1+I_2,\ \text{say.}
\end{aligned}\qquad\ldots(39)$$

By Leibniz theorem, Theorem 6 and the hypothesis that (i) holds, it can be easily seen that

$$I_1=O(\lambda_n^{-\tau/2}).\qquad\ldots(40)$$

Let $x(u)$ denote the characteristic function of the interval $[a_1^*,b_1^*]$. Then, by Leibniz theorem and Theorem 5 there follows

$$I_2=\left\|-\sum_{i=0}^{m-1}\binom{m}{i}g^{(m-i)}(t)\,S\lambda_n^{(i)}(f,k,t)+S_{\lambda_n}^{(m)}(f(u)-g(u)-g(t))x(u),k,t)\right\|_{C[a_2^*,b_2^*]}$$

$$=\left\|I_3+I_4\right\|_{C[a_2^*,b_2^*]}+O\left(\lambda_n^{-(k+1)}\right),\text{say.}\qquad\ldots(41)$$

The estimate of I_3 can be made immediately by Theorem 6:

$$I_3=-\sum_{i=0}^{m-1}\binom{m}{i}\; g^{(m-i)}(t)f^{(i)}(t)+O(\lambda_n^{-\tau/2})\qquad\ldots(42)$$

uniformly in $t\in\left[a_2^*,b_2^*\right]$.

Since $f^{(m+p-1)}$ exists on $[a_1^*,b_1^*]$, by a Taylor expansion of f and Theorem 5 we get

$$I_4=\sum_{j=0}^{k}C(j,k)\sum_{i=0}^{m+p-1}\frac{f^{(i)}(t)}{i!}\int_A^B W^{(m)}(d_j\lambda_n,t,u)(u-t)^i(g(u)-g(t))du$$

(*equation continued on p.* 487)

$$+\frac{1}{(m+p-1)!}\sum_{j=0}^{k} C(j,k)\int_A^B W^{(m)}(d_j\lambda_n,t,u)(f^{(m-p-1)}(\xi)-f^{(m-p-1)}(t))$$

$$\times(u-t)^{m+p-1}(g(u)-g(t))\,x(u)du+o(\lambda^{-(k+1)})(\xi \text{ lying between } u \text{ and } t)$$

$$=I_5+I_6+o(\lambda_n^{-(k+1)}), \text{ say,} \quad ...(43)$$

where the o-term is uniform for $t\in\left[a_2^*, b_2^*\right]$.

Now, substituting the Taylor expansion

$$g(u)=\sum_{r=0}^{m+p+1}\frac{g^{(r)}(t)}{r!}(u-t)^r+\in(u,t)(u-t)^{m+p+1} \quad ...(44)$$

where $\in(u,t)\to 0$ as $u\to t$, in I_5, by Theorem 5 we obtain the dominated part of I_5 as $\sum_{r=1}^{m}\binom{m}{r}g^{(r)}(t)f^{(m-r)}(t)+O\left(\lambda_n^{-(k+1)}\right)$ uniformly in $t\in\left[a_2^*,b_2^*\right]$. On the other hand, by Lemmas 2 and 3 and Schwraz inequality the remaining part of I_5 is easily seen to be $O\left(\lambda_n^{(p+1)/2}\right)=O\left(\lambda_n^{-\tau/2}\right)$ uniformly in $t\in\left[a_2^*,b_2^*\right]$.

Thus,

$$I_5=\sum_{r=1}^{m}\binom{m}{r}g^{(r)}(t)f^{(m-r)}(t)+O\left((\lambda_n^{\tau/2}\right), \quad ...(45)$$

uniformly in $t\in\left[a_2^*, b_2^*\right]$.

Lastly, by Lemmas 2 and 3, mean value theorem and Hölder's inequality we obtain

$$\|I_6\|_{C[a_2^*,b_2^*]}=O\left(\lambda_n^{-(p+1-\delta)/2}\right)=O\left(\lambda_n^{-\tau/2}\right) \quad ...(46)$$

by choosing δ such that $0<\delta\leqslant p+1-\tau$.

Combining the above estimates we conclude that

$$\left\|S\lambda_n^{(m)}(fg,k,t)-(fg)^{(m)}(t)\right\|_{C[a_2^*,b_2^*]}\ O(\lambda_n^{-\tau/2}).$$

Since supp $fg\subset\left(a_2^*,b_2^*\right)$ therefore by Lemmas 4 and 5 $(fg)^{(m)}\in \text{Liz}\,(\alpha,k+1;a_2^*b_2^*.)$ From which (ii) is immediate since $g(t)=1$ on $\left[a_2,b_2\right]$ and $\left[a_2,b_2\right]\subset\left(a_2^*,b_2^*\right)$.

4. Proof of the Saturation Theorem

First, let us assume (i). It is clear that $f^{(m)}\in C[a_1,b_1]$. Also, (i) $\Rightarrow$ (iii) of Theorem 1 yields that $f^{(2k+m+1)}$ exists and is continuous on (a_1,b_1). Choose $a_1^*,b_1^*,a_2^*b_2^*$ in such a way that $a_1<a_1^*<a_2^*<a_2<b_2<b_2^*<b_1^*<b_1$. Then, there exists a function $f^*\in C_0^{2k+m+1}(a_1,b_1)$ such that $f^*\equiv f$ on $\left[a_1^*,b_1^*\right]$. By Theorem 5 it follows that

$$\left\| S\lambda_n^m(f^*,k,t)-f^{*(m)}(t)\right\|_{C[a_2^*,b_2^*]} = O\left(\lambda_n^{-(k+1)}\right). \quad ...(47)$$

Then, for each $q \in C_0^m\left[a_1,b_1\right]$ and $g\in C_0^\infty\left(a_2^*,b_2^*\right)$, by Theorem 3 we have

$$\lambda^{k+1}\left\langle S\lambda^{(m)}(q,k,t)-q^{(m)}(t),g(t)\right\rangle$$

$$=\lambda^{k+1}\left\langle S\lambda(q,k,t)-q(t),\ g^{(m)}(t)\right\rangle$$

$$=\lambda^{k+1}\left\langle q(u),\sum_{j=9}^{k} C(j,k)\sum_{i=1}^{k+1}\frac{g_{i,k+1}^{[m]}(u)}{(d_j\lambda)^i}+o\left(\lambda^{-(k+1)}\right)\right\rangle$$

$$=\left\langle q(u),\sum_{j=0}^{k} C(j,k)\frac{g_{k+1,k+1}^{[m]}(u)}{d_j^{k+1}}+o\ (1)\right\rangle\cdot$$

where $g_{i,k+1}^{[m]} \in C_0^\infty\left(a_2^*,b_2^*\right)$, $i=1,2,\ldots,k+1$ are functions, depending on g and the o-term holds uniformly in $[a_1,b_1]$.

Hence, for all λ sufficiently large

$$\lambda^{k+1}\left|\left\langle S_\lambda^{(m)}(q,k,t)-q^{(m)}(t),g(t)\right\rangle\right| \leqslant M\,\|q\|_{[Ca_1,b_1]} \quad ...(48)$$

where M is independent of λ and q_*.

Since $f^* \in C_0^{2k+m+1}(a_1,b_1)$ there exists a sequence $\left\{f_\sigma \in C_0^{2k+m+2}(a_1,b_1)\right\}$ such that as $\sigma\to\infty$, $\|f_\sigma-f^*\|_{C[a_1,b_1]}\to 0$. Then, by Theorem 5, for any $g\in C_0^\infty(a_1,b_1)$ and each function f_σ we get

$$\lim_{\lambda\to\infty}\lambda^{k+1}\left\langle S_\lambda^{(m)}(f_\sigma,k,t)-f_\sigma^{(m)}(t),g(t)\right\rangle$$

$$=\left\langle\sum_{j=m}^{2k+m+2} Q(j,k,m,t)f_\sigma^{(j)}(t),g(t)\right\rangle$$

$$=\left\langle f_\sigma(t),\sum_{j=1}^{2k+m+2} Q^*(j,k,m,t)\,g^{(j)}(t)\right\rangle,$$

where $Q^*_{2k+m+2}(D)=\sum_{j=1}^{2k+m+2} Q^*(j,k,m,t)\,D^j$ denotes the differential operator adjoint to

$Q_{2k+m+2}(D)=\sum_{j=m}^{2k+m+2} Q(j,k,m,t)\,D^j.$

From (47) it is clear that there exists a subsequence $\left\{\lambda_{nq}^{k+1}\left[S_{\lambda nq}^{(m)}(f^*,k,t)-f^{*(m)}(t)\right]\right\}_{q=1}^{\infty}$ which converges to a function $h\in L_\infty\left[a_2^*,b_2^*\right]$

in the weak*-topology of the space $L_\infty\left[a_2^*,b_2^*\right]$. Thus, for every $g\in C_0^\infty\left(a_2^*,b_2^*\right)$ we have

$$\lim_{q\to\infty}\lambda_{nq}^{k+1}\left\langle S_{\lambda_{nq}}^{(m)}(f^*,k,t)-f^{*(m)}(t),g(t)\right\rangle=\left\langle h(t),g(t)\right\rangle. \quad \ldots(50)$$

By (48), it follows that

$$\lim_{q\to\infty}\lambda_{n_q}^{k+1}\left|\left\langle S_{\lambda_{nq}}^{(m)}(f^*-f_\sigma,k,t)-(f^*-f_\sigma)^{(m)}(t),g(t)\right\rangle\right|$$
$$\leqslant M\left\|f^*-f_\sigma\right\|_{C[a1,b1]}. \quad \ldots(51)$$

Combining (49-51) we get

$$\left\langle f^*(t),Q_{2k+m+2}^*(D)g\right\rangle=\lim_{\sigma\to\infty}\left\langle f_\sigma(t),Q_{k+m+2}^*(D)g\right\rangle$$
$$=\lim_{\sigma\to\infty}\Big[\lim_{q\to\infty}\lambda_{n_q}^{k+1}\left\langle S_{\lambda_{nq}}^{(m)}(f^*-f_\sigma,k,t)-(f^*-f_\sigma)^{(m)}(t),g(t)\right\rangle$$
$$+\left\langle f_\sigma(t),Q_{2k+m+2}^*(D)\,g\right\rangle\Big]=\left\langle h(t),g(t)\right\rangle.$$

This implies

$$Q_{2k+m+2}(D)f^*=h(t), \quad \ldots(52)$$

as generalized function

Now, by Theorem 5, $Q(2k+m+2,k,m,t)\neq 0$, therefore regarding (52) as a first order linear differential equation for $f^{*(2k+m-1)}$, we conclude that $f^{*(2k+m+1)}\in A.C.\left[a_2^*,b_2^*\right]$ and hence that $f^{*(2k-m+2)}\in L_\infty\left[a_2^*,b_2^*\right]$. From this, (ii) is immediate, since $\left[a_2 b_2\right]\subset\left[a_2^*,b_2^*\right]$ and $f^*\equiv f$ on $\left[a_2,b_2\right]$.

Also, (iii)⇒ (iii) is immediate from Theorem 6.

To prove (iv) ⇒ (v), assuming (iv) and proceeding as in the proof of the implication (i) ⇒ (ii) we get $Q_{2k+m+2}(D)\,f^*(t)=0$, from which (v) is clear.

Finally, (v) ⇒ (vi) follows from Theorem 5.

This comples the proof of the saturation theorem.

5. Concluding Remarks

Theorems 1 and 4-6 are applicable to the Bernstein polynomials Szász operators, Baskakov operators, Post-Widder operators and Gauss-Weierstrass integrals which are of exponential type as shown by May (1976). Moreover, Theorems 2 and 3 are valid for the Post-Widder operators and Gauss-Weierstrass integrals which are also regular.

By using other methods we have shown that the saturation theorem also holds for several non-regular exponential type operators including the Bernstein polynomials, Szász operators and the Baskakov operators. These results will appear elsewhere.

Finally it is remarked that in the results of this paper, the domain $C(A,B)$ can be replaced by $C\psi(A,B)$ as defined by May (1976) where ψ is a growth test function for $\{S_\lambda,\lambda\in\Lambda\}$. The details of the proofs for this general situation are given by Agrawal (1978).

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