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# INEQUALITIES FOR CERTAIN CONFLUENT HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

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We use Eulerian integral representation technique suggested by Luke and series manipulation using Flett, Carlson and Buschman's results in obtaining two sided inequalities for certain confluent hypergeometric functions of two variables, viz., $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Psi_1$, $\Psi_2$, $\Xi_1$, $\Xi_2$. Specific numerics that are given not only give credence to the various formulas that are presented but also point out the utility of these results from a computational stand point. The inequalities obtained have their own significance since their evaluation often takes much less effort than the evaluation of a double infinite series.

## 1. INTRODUCTION

The concepts of Páde approximation, convexity, mean values and that of a determinant with dominant main diagonal have been utilized by Luke (1972), Flett (1972), Carlson (1966) and Buschman (1976) respectively in developing inequalities for generalized hypergeometric functions of one variable. Whereas Luke (1974) has further exploited his results in obtaining bounds for Appell functions $F_1$, $F_2$ and $F_3$ using their Eulerian integrals. We have used Flett, Carlson and Buschman's results in obtaining some further inequalities for $F_1$ and $F_2$ through series manipulations (see Joshi and Arya 1980). Interestingly our results are complementary to those of Luke in the sense that they hold for different domains of validity. In this note these notions are further exploited in obtaining two sided inequalities for certain confluent hypergeometric functions viz., $\Phi_1$, $\Phi_2$, $\Phi_3$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $\Xi_1$, $\Xi_2$. Since the definitions of the functions $\Phi_1$, $\Phi_2$ and $\Xi_2$ as given in Erdélyi *et al.* (1953) appear to be erroneous, we cite the definitions as they are given in Appell and Kampé de Fériet (1926). In what follows, it shall be understood that all the parameters and variables are real numbers.

## 2. INEQUALITIES FOR $\Phi_1$

We show that the function $\Phi_1$, defined by

$$\Phi_1(a;b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m x^m y^n}{(c)_{m+n} m!n!}, \quad |x| < 1, \tag{2.1}$$

$c$ being not a negative, integer or zero, and having the integral representation

$$\Phi_1(a;b;c;-x,-y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1+tx)^{-b} e^{-yt} dt, \; c>a>0, \quad \ldots(2.2)$$

admits the following inequality which is sharp enough for small values of $y$. In other words, for small values of $y$, the lower and upper bounds differ only marginally.

*Theorem* 1—If $c>a>0$, $0<b\leqslant 1$, $0<x<1$, $y>0$, $2bx>y$, $(1-b)x>2y$, then

$$\frac{2bx+y}{2bx-y}\left(1+\frac{abx}{c}\right)^{-1} - \frac{2y}{(2bx-y)}\left[\frac{c-a}{c(1+a)} + \frac{a(1+c)}{c(1+a)}\left(1+\frac{(1+a)y}{2(1+c)}\right)^{-1}\right]$$
$$<\Phi_1(a;b;c;-x,-y) < \frac{((1-b)x-2y)}{((1+b)x-2y)}\left[\frac{c-a}{c(1+a)} + \frac{a(1+c)}{c(1+a)}\left(1+\frac{(1+a)y}{(1+c)}\right)^{-1}\right]$$
$$+\frac{2bx}{((1+b)x-2y)}\left[\frac{c-a}{c(1+a)} + \frac{a(1+c)}{c(1+a)}\left(1+\frac{(1+a)(1+b)x}{2(1+c)}\right)^{-1}\right]. \quad \ldots(2.3)$$

To prove, consider the inequality [Luke 1972, (5.3)]

$$-1+2\left(1+\frac{az}{2c}\right)^{-1} < {}_1F_1(a;c;-z) < \frac{c-a}{c(1+a)} + \frac{a(1+c)}{c(1+a)}\left(1+\frac{(1+a)z}{(1+c)}\right)^{-1},$$
$$z>0, c\geqslant a>0. \quad \ldots(2.4)$$

For $a=c$, it gives rise to [see also Luke 1975, 3.24 (3)]

$$-1+2(1+z/2)^{-1} < e^{-z} < (1+z)^{-1}, \; z>0, \quad \ldots(2.5)$$

which is sharp enough for $z$ being very small. In fact the left inequality is very much nearer to the actual value than the right inequality and for $z=0$, it reduces to equality.

Application of the inequality [Luke 1972, (4.4)]

$$(1+\beta z)^{-1} < (1+z)^{-\beta} < \frac{1-\beta}{1+\beta} + \frac{2\beta}{1+\beta}\left(1+\left(\frac{1+\beta}{2}\right)z\right)^{-1}; \; 0<\beta\leqslant 1, \quad \ldots(2.6)$$

along with (2.5) and the Euler integral representation of ${}_2F_1$ [Erdélyi *et al.* 1953, (2.1.3)] in (2.2) gives

$$\frac{(2bx+y)}{(2bx-y)}\,{}_2F_1(1,a;c;-bx) - \frac{2y}{(2bx-y)}\,{}_2F_1\left(1,a;c;-\frac{y}{2}\right) < \Phi_1(a;b;c;-x,-y)$$
$$< \frac{2bx}{((1+b)x-2y)}\,{}_2F_1\left(1,a;c;-\left(\frac{1+b}{2}\right)x\right) + \frac{((1-b)x-2y)}{((1+b)x-2y)}\,{}_2F_1(1,a;c;-y), \quad \ldots(2.7)$$

under the conditions as stated in the theorem.

Now appropriate application of [8; (4.6)]

$$\left(1+\frac{az}{c}\right)^{-1} < {}_2F_1(1,a;c;-z) < \frac{c-a}{c(1+a)} + \frac{a(1+c)}{c(1+a)}\left(1+\frac{(1+a)z}{(1+c)}\right)^{-1},$$
$$z>0, c\geqslant a>0 \quad \ldots(2.8)$$

to the ${}_2F_1$'s in (2.7) then gives the required theorem.

In order to have corresponding theorem for positive arguments for $0<x<1/2$, one will have to use the transformation formula

$$\Phi_1(a;b;c;x,y) = e^{y}(1-x)^{-b}\,\Phi_1(c-a;b;c;-x/(1-x),-y), \quad \ldots(2.9)$$

which is a consequence of Kummer's and Euler's transformation formulas [Erdélyi *et al.* 1953, (6.3.7), (2.10.6)] in the series representation of $\Phi_1$.

Another inequality for positive real arguments can also be obtained by Flett's theorem (1972), part of which can be rewritten as

$$(1-x)^{c-a-b} < {}_2F_1(a,b;c;x) < \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b},\ c>a>c-b>0,\ 0<x<1, \quad ...(2.10)$$

which will be reversed if $a>c>b>0$, $0<x<1$.
Indeed, writing (2.1) in the form

$$\Phi_1(a;b;c;x,y) = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{(c)_n n!}\, {}_2F_1(a+n,\ b;\ c+n;\ x) \quad ...(2.11)$$

applying (2.10) and approximating the resulting ${}_1F_1$ by [Luke 1972, (5.8)]

$$1+\frac{az}{b}\exp\left(\frac{1+a}{1+b}\right)\frac{z}{2} < {}_1F_1(a;b;z) < 1+\frac{az}{b}\left(1-\frac{(1+a)}{2(1+b)}+\frac{(1+a)}{2(1+b)}e^z\right),$$

$z>0, b\geqslant a>0$ or by [Luke 1972, (5.6)] ...(2.12)

$e^{ax/b} < {}_1F_1(a;b;x) < 1+a/b(e^x-1),\ b\geqslant a>0, x\neq 0,$ ...(2.13)

or by transforming appropriately (2.4) by recourse to Kummer's formula [Erdélyi *et al.* 1953, (6.3.7)], we have the theorem

*Theorem* 2—If $c>a>c-b>0$, $b>0$, $0<x<1$, $y>0$, then

$$(1-x)^{c-a-b}\max\left\{e^{ay/c}, 1+\frac{ay}{c}\exp\left(\frac{1+a}{1+c}\right)\frac{y}{2}, e^y\left[2\left(1+\frac{(c-a)y}{2c}\right)^{-1}-1\right]\right\}$$

$$<\Phi_1(a;b;c;x,y) < \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}e^y(1-x)^{c-a-b}. \quad ...(2.14)$$

Note that the right inequality in (2.14) is far from sharp, the left inequality on the other hand is nearer to the actual value.

As a numerical verification, we have from theorem 1

$.8718905 < \Phi_1(.3;\ .8;\ .7;\ -.4,\ -.02) < .8856394,$ ...(2.15)

and from Theorem 2

$1.1186975 < \Phi_1(.4;\ .6;\ .8;\ .1,\ .02) < 1.6598084.$ ...(2.16)

## 3. Inequalities for $\Phi_2$

In the Euler integral representation for $\Phi_2$

$$\Phi_2(b_1,b_2;c;-x,-y) = \Gamma\begin{bmatrix} c \\ b_1, b_2, c-b_1-b_2\end{bmatrix}\int_0^1\int_0^1 t^{b_2-1}(1-t)^{c-b_2-1}u^{b_1-1}$$

$$.(1-u)^{c-b_1-b_2-1}\exp-(yt+ux(1-t))\,du\,dt,\ c>b_1+b_2,\ b_1>0, b_2>0, \quad ...(3.1)$$

whose double series representation is given by

$$\Phi_2(b_1,b_2;c;x,y) = \sum_{m,n=0}^{\infty}\frac{(b_1)_m(b_2)_n\, x^m y^n}{(c)_{m+n}\, m!\, n!}, \quad ...(3.2)$$

$c$ being not zero or a negative integer.

Appropriately applying (2.5) and Euler integral representations of ${}_2F_1$, ${}_1F_1$ and the beta function [Erdélyi 1953, (2.1.3), (6.5.1), (1.5.1)], and the inequalities (2.8) and (2.4) one obtains

*Theorem* 3—If $y \geqslant x > 0$, $c > b_1 + b_2$, $b_1 > 0$, $b_2 > 0$, then

$$-1+2\left(1+\frac{b_1x+b_2y}{2c}\right)^{-1} < \Phi_2(b_1,b_2;c;-x,-y) < \frac{c-b_1-b_2}{c(1+b_1)(1+b_2)} + \frac{b_1(1+c-b_2)}{c(1+b_1)(1+b_2)}$$
$$\times\left[1+\frac{(1+b_1)x}{1+c-b_2}\right]^{-1} + \frac{b_2(1+c)}{c(1+b_1)(1+b_2)(c-b_2)}\left[1+\frac{(1+b_2)y}{1+c}\right]^{-1}$$
$$\times\left\{c-b_1-b_2+b_1(1+c-b_2)\left[1+\frac{(1+b_1)(c-b_2)x}{(1+c-b_2)(1+c+(1+b_2)y)}\right]^{-1}\right\}. \quad \ldots(3.3)$$

Due to the presence of exponential term in the integral representation of $\Phi_2$ as given by (3.1), the above inequality will be sharp only for values of $x$ and $y$ in the neighbourhood of zero. Also the exact value of the function will lie to the left of the mean value of the lower and upper bounds.

Examining now the symmetry aspect of $\Phi_2$ which is exhibited by

$$\Phi_2(b_1,b_2;c;x,y) = \Phi_2(b_2,b_1;c;y,x), \quad \ldots(3.4)$$

and states that $\Phi_2$ is unchanged if $b_1 \leftrightarrow b_2$ and $x \leftrightarrow y$ simultaneously. Whereas the left inequality of (3.3) is symmetrical in the sense of (3.4), this is not for the right side. Thus if the inequality with $x \geqslant y > 0$ is desired, in (3.3) $b_1$ and $x$ may be replaced by $b_2$ and $y$ and vice versa. Similar observation may be made for the inequality that would be obtained from the limiting case of the integral representation for $F_1$ given by Luke [1974, (1)].

Another theorem, giving two inequalities symmetric in sense of (3.4), one for positive arguments and another for negative arguments, that follows from the integral

$$\Phi_2(b_1,b_2;c;x,y) = \Gamma\left[\begin{matrix} c \\ b_2, c-b_2 \end{matrix}\right] \int_0^1 t^{c-b_2-1}(1+t)^{b_2-1}e^{y}(1-t)$$
$$\times {}_1F_1(b_1;c-b_2;xt)dt; \; c > b_2 > 0, \quad \ldots(3.5)$$

and the inequalities (2.13) and [Luke 1972, (5.3)]

$$e^{-az/c} < {}_1F_1(a;c;-z) < 1-a/c\,(1-e^{-z}); z>0, \; c \geqslant a > 0, \quad \ldots(3.6)$$

respectively, is given by :

*Theorem* 4—If $c > b_1 + b_2$, $b_1 > 0$, $b_2 > 0$, $b_1x > (c-b_2)y$ or $b_2y > (c-b_1)x$, then

$$\text{(a)} \; e^{(b_1x+b_2y)/c} < \Phi_2(b_1,b_2;c;x,y) < \frac{(c-b_1-b_2)}{c} + \frac{b_1e^x}{c} + \frac{b_2e^y}{c}. \quad \ldots(3.7)$$

$$\text{(b)} \; e^{-(b_1x+b_2y)/c} < \Phi_2(b_1,b_2;c;-x,-y) < \frac{c-b_1-b_2}{c} + \frac{b_1e^{-x}}{c} \; \frac{b_2e^{-y}}{c}. \quad \ldots(3.8)$$

Numerically, theorem 3 gives

$$.794871795 < \Phi_2(.4, .2; .7; -.2, -.4) < .819513724, \quad \ldots(3.9)$$

while Theorem 4 gives

$$1.256803293 < \Phi_2(0.4, 0.2; 0.7; 0.2, 0.4) < 1.2670372, \quad \ldots(3.10)$$
$$0.795669462 < \Phi_2(0.4, 0.2; 0.7; -0.2, -0.4) < 0.80222233. \quad \ldots(3.11)$$

## 4. Inequalities for $\Psi_1$

For the function $\Psi_1$, defined by

$$\Psi_1(a; b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\, x^m\, y^n}{(c)_m\,(c')_n\, m!\, n!}, \quad |x| < 1, \qquad \ldots(4.1)$$

where we naturally suppose that $c$ and $c'$ are not negative integers or zero, the following inequalities are established :

*Theorem* 5—If $0<a \leqslant 1$, $0<a<c'$, $0<b<c$, $0<x<1$, $y>0$, $a^2 y>2c'(1-a)$ $(1+a)^2 y> (1+c')(1-a)$, then

$$\frac{2ay}{(a^2y-2c'(1-a))}\left[1+\frac{bx}{c}+\frac{ay}{2c'}\right]^{-1} - \frac{a^2y+2c'(1-a)}{a^2y-2c'(1-a)}\left\{\frac{c-b}{c(1+b)}+\frac{b(1+c)}{c(1+b)}\right.$$
$$\left.\left[1+\frac{a(1+b)x}{(1+c)}\right]^{-1}\right\} < \Psi_1(a; b; c, c'; -x, -y) < \frac{1-a}{1+a}-\frac{a(1-a)y}{c'(1+a)}$$
$$\left[1+\frac{bx}{c}+\frac{(1+a)y}{(1+c')}\right]^{-1}+\frac{2a(c'-a)}{cc'(1+b)(1+a)^2}$$
$$\times\left\{c-b+b(1+c)\left[1+\frac{(1+a)(1+b)x}{2(1+c)}\right]^{-1}\right\}+\frac{4a^2(1-c')^2y[(1+a)^2y-(1+c')(1-a)]^{-1}}{c'(1+a)(1+c'+(1+a)y)}$$
$$\times\left\{\frac{c-b}{c(1+b)}+\frac{b(1+c)}{c(1+b)}\left[1+\frac{(1+b)(1+c')x}{(1+c)(1+c'+(1+a)y)}\right]^{-1}\right\}$$
$$-\frac{2a^2(1-a)(1+c')^2}{c'(1+a)^3((1+a)^2y-(1+c')(1-a))}\left[1+\frac{b(1+a)x}{2c}\right]^{-1}. \qquad \ldots(4.2)$$

*Theorem* 6—If $c>a>c-b>0$, $c'>a$, $b>0$, $1 \geqslant x>0$, $y>0$, then

$$(1-x)^{c-a-b}\max\left\{\begin{array}{l} e^{ay/c'(1-x)},\ e^{y/(1-x)}\left[2\left(1+\dfrac{(c'-a)y}{2c'(1-x)}\right)^{-1}-1\right], \\ 1+(ay/c'(1-x))\, e^{(1+a)y/(1+c')(1-x)} \end{array}\right.$$
$$< \Psi_1(a; b; c, c'; x, y) < \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b} \times$$
$$\times \min\left\{\begin{array}{l} 1+\left(\dfrac{a+b-c}{c'}\right)(e^{y/(1-x)}-1),\ 1+\dfrac{(a+b-c)y}{c'(1-x)}\left[1+\dfrac{(1+a+b-c)}{2(1+c')}(e^{y/(1-x)}-1)\right], \\ e^{y/(1-x)}\left\{\dfrac{a+b-c}{c'(1+c+c'-a-b)}+\dfrac{(c+c'-a-b)(1+c')}{c'(1+c+c'-a-b)}\right. \\ \left.\left[1+\dfrac{(1+c+c'-a-b)y}{(1+c')(1-x)}\right]^{-1}\right\}. \end{array}\right. \qquad \ldots(4.3)$$

*Theorem* 7—If $a>c>b>0$, $c'>a$, $0<x<1$, $y>0$, then

$$\frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}\max\left\{\begin{array}{l} \left(\exp\dfrac{(a+b-c)y}{c'(1-x)}\right),\ \left(\exp\dfrac{y}{1-x}\left[2\left(1+\dfrac{(c+c'-a-b)y}{2c'(1-x)}\right)^{-1}\right.\right. \\ \qquad\qquad\qquad\qquad \left.-1\right], \\ 1+\dfrac{(a+b-c)y}{c'(1-x)}e^{(1+a+b-c)y/2(1+c')(1-x)} \end{array}\right.$$
$$< \Psi_1(a; b; c, c'; x, y) < (1+x)^{c-a-b}$$
$$\times\min\left\{\begin{array}{l} 1+\dfrac{a}{c'}(e^{y/(1-x)}-1),\ 1+\dfrac{ay}{c'(1-x)}\left[1+\dfrac{(1+a)}{2(1+c')}(e^{y/(1-x)}-1)\right], \\ \dfrac{e^{y/(1-x)}}{c'(1+c'-a)}\left[a+(c'-a)(1+c')(1+(1+c'-a)y/(1+c')(1-x))^{-1}\right]. \end{array}\right. \qquad \ldots(4.4)$$

*Theorem* 8—If $c'>a>c+1$, $c>b>0$, $-\infty<x<1$, $x\neq 0$, $y>0$, then

$$(1-x)^{c-a-b}\,(1-x+(bx/c))^{a-c}$$

$$.\max\begin{cases}\exp\dfrac{ay\ (1-x+(bx/c))}{c'(1-x)},\ 1+\dfrac{ay(1-x+(bx/c))}{c'(1-x)}\exp\dfrac{(1+a)y(1-x+(bx/c))}{2(1+c')(1-x)},\\ \exp\dfrac{y\,(1-x+(bx/c))}{(1-x)}\left[\ 2\left[1+\dfrac{(c'-a)y(1-x+(bx/c))}{2c'\,(1-x)}\right]^{-1}-1\right]\end{cases}$$

$$<\Psi_1\,(a;\,b;c,c';x,y)$$

$$<\frac{b}{c}\ (1-x)^{c-a-b}\min\begin{cases}1+\dfrac{a}{c'}\ (e^{y/(1-x)}-1),\ 1+\dfrac{ay}{c'(1-x)}\left[\ 1+\dfrac{(1+a)}{2+(1-c')}\right.\\ \qquad\qquad \left.(e^{y/(1-x)}-1)\right],\\ \dfrac{e^{y(1-x)}}{c'(1+c'-a)}\left[a+(c'-a)\ (1+c')\left(1+\dfrac{(1+c'-a)y}{(1+c')(1-x)}\right)^{-1}\right.\end{cases}$$

$$+\left(1-\frac{b}{c}\right)(1-x)^{-b}\min\begin{cases}1+\dfrac{a}{c'}\ (e^{y}-1),\ 1+\dfrac{ay}{c'}\left(1+\dfrac{(1+a)}{2(1+c')}(e^{y}-1)\right),\\ e^{y}\left[\dfrac{a}{c'(1+c'-a)}+\dfrac{(c'-a)(1+c')}{c'(1+c'-a)}\left(1+\dfrac{(1+c'-a)y}{(1+c')}\right)\right]^{-1}.\end{cases}\qquad\ldots(4.5)$$

The proof of Theorem 5 is very much akin to that of Theorems 1 and 3. Indeed, the integral representation

$$\Psi_1\,(a;\,b;c,\,c';-x,-y)=\Gamma\left[\begin{matrix}c\\ b,c-b\end{matrix}\right]\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1+tx)^{-a}$$

$$\times\ {}_1F_1\,(a;\,c';\,-y/(1+tx))\,dt;\quad c>b>0,\qquad\ldots(4.6)$$

with appropriate applications of (2.4), (2.6), the Euler integral representations of ${}_2F_1$ and the beta functions [Erdélyi 1953, (2.1.3), (1.5.1)] and the inequality (2.8) will yield the theorem.

For the proofs of Theorems 6 and 7, express $\Psi_1$ as

$$\Psi_1\,(a;\,b;\,c,\,c';\,x,\,y)=\sum_{n=0}^{\infty}\frac{(a)_n\,y^n}{(c')_n\,n!}\ {}_2F_1\,(a+n;\,b;\,c';\,x)\,,\qquad\ldots(4.7)$$

apply (2.10) appropriately and estimate the resulting ${}_1F_1$'s either by (2.12) or (2.13) or by (2.4) after transforming it by Kummer's first formula. Theorem 8 may also be proved similarly as Theorems 6 and 7 but using Carlson's theorem (1972, Th. 2); see also Joshi and Arya (1980 Th. 2).

Some other inequalities for $\Psi_1$ could also be obtained using Buschman's theorems (1976, Th. 1, Th. 2) in terms of hypergeometric functions of one variable. The details are however omitted.

These theorems could be checked for special values of parameters and variables. For example, from Theorem 5, we have

$0.4433205 < \Psi_1\,(0.9;\,0.2;\,0.4,\,0.1;\,-0.3,\,-0.8) < 0.50229464$,

and from Theorem 6, $1.17520187 < \Psi_1\,(0{\cdot}6;\,0.4;\,0.7,\,0.8;\,0.2,\,0.1) < 1.18170178$.
However for Theorems 7 and 8, considering the values for parameters and variables from the common domain of validity, from Theorem 7 one obtains

$1.4186866 < \Psi_1(1.5; 0.2; 0.4, 1.6; 0.5, 0.1) < 2.97311078$, and from Theorem 8
$2.0655459 < \Psi_1(1.5; 0.2; 0.4, 1.6; 0.5, 0.1) < 2.11747238$.
Comparison shows that in the common domain of validity, Theorem 8 gives better estimates than Theorem 7.

## 5. Inequalities for $\Xi_1$

The inequality proved is:

*Theorem* 9—If $c>a+a'$, $a>0$, $a'>0$, $1\geqslant b>0$, $1>x>0$,

$$\frac{a'y}{2(c-a)} < \frac{bx}{1+bx},\ \min\left\{\frac{2(c-a)}{a'},\ \frac{1+c-a}{1+a'}\right\} > y > 0, \text{ then}$$

$$\frac{2a'y(1+(a'y/2c))^{-1}}{2bx(c-a)+a'y(1+bx)} + \frac{2bx(c-a)-a'y(1+bx)}{2bx(c-a)+a'y(1+bx)}\left[1+\frac{abx}{c}\right]^{-1}$$
$$< \Xi_1(a,a';b;c;-x,-y) < \frac{(1-b)(c-a-a')}{(1+b)(1+a')(c-a)} + \frac{a'(1-b)}{c(c-a)(1+a')(1+b)}$$
$$\left\{a+(c-a)(1+c)\times\left[1+\frac{(1+a')y}{(1+c)}\right]^{-1}\right\} + \frac{2b(c-a-a')}{(1+b)(1+a')(c-a)}\left\{\frac{c-a}{c(1+a)}\right.$$
$$\left. + \frac{a(1+c)}{c(1+a)}\left[1+\frac{(1+a)(1+b)x}{2(1+c)}\right]^{-1}\right\} + \frac{2a'b(1+c-a)}{(1+b)(1+a')(c-a)}\left[\frac{(1+b)x}{2}+\frac{(1+a')y}{(1+c-a)}\right.$$
$$\left. + \frac{(1+a')(1+b)xy}{2(1+c-a)}\right]^{-1}\left\{\frac{(1+b)x}{2c(1+a)}\times\left(c-a+a(1+c)\left[1+\frac{(1+a)(1+b)x}{2(1+c)}\right]^{-1}\right)\right.$$
$$\left. + \frac{(1+a')y}{c(1+c-a)^2}\left(a+(c-a)(1+c)\left[1+\frac{(1+a')y}{(1+c)}\right]^{-1}\right)\right\}, \qquad \ldots(5.1)$$

where $\Xi_1$ has the double series representation

$$\Xi_1(a,a';b;c;-x,-y) = \sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(-x)^m(-y)^n}{(c)_{m+n}\, m!\, n!},\ |x|<1, \qquad \ldots(5.2)$$

and $c$ is not a negative integer or zero.

The proof follows from the integral representation

$$\Xi_1(a,a';b;c;-x,-y = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}$$
$$\times (1+tx)^{-b}\, {}_1F_1(a';c-a;-y/(1-t))\, dt;\ c>a>0, \qquad \ldots(5.3)$$

in the manner of Theorems 1, 3 and 6.

Note that if $\dfrac{bx}{1+bx} < \dfrac{a'y}{2(c-a)}$ in place of $\dfrac{bx}{1+bx} > \dfrac{a'y}{2(c-a)}$

the left member of the inequality will assume the form

$$\frac{2a'y[1+(a'y/2c)]^{-1}}{2bx(c-a)+a'y(1+bx)} - \frac{(a'y(1+bx)-2(c-a)bx)}{(a'y(1+bx)+2bx(c-a))}\left\{\frac{c-a}{c(1+a)}+\frac{a(1+c)}{c(1+a)}\right.$$
$$\left.\left[1+\frac{(1+a)bx}{(1+c)}\right]^{-1}\right\}, \qquad \ldots(5.4)$$

and the right member will remain unchanged.

As a numerical verification, we have from Theorem 9

$$.8674671 < \Xi_1(.2,\ .3;\ .2;\ .8;\ -.6,\ -.3) < .9391287, \quad \ldots(5.5)$$

while from the other form of the theorem, as given by (5.4)

$$.7585274 < \Xi_1(.2,\ .5;\ .2;\ .8;\ -.5,\ -.4) < .76488095. \quad \ldots(5.6)$$

## 6. Inequalities for $\Phi_3$ and $\Xi_2$

It is shown that the functions $\Phi_3$ and $\Xi_2$ satisfy the inequalities:

*Theorem* 10—If $c>1/2$, $c>b>0$, $x>0$, $y>0$, then

$$e^{bx/c} < \Phi_3(b;c;x,y) < \tfrac{1}{2}\{\exp(-2\sqrt{y})+(\exp(2\sqrt{y})\}\ (1+b/c\ (e^x-1)) \quad \ldots(6.1)$$

where the function $\Phi_3$ is defined by

$$\Phi_3(b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(b)_m x^m y^n}{(c)_{m+n} m!n!} \quad \ldots(6.2)$$

$c$ is not a negative integer or zero.

*Theorem* 11—If $a>c+1>3/2$, $c>b>0$, $0<x<1$, $y>0$, then

$$(1-x)^{c-a-b}\ (1-x+(bx/c))^{a-c} < \Xi_2(a;b;x,y)$$

$$< \tfrac{1}{2}\{\exp(2\sqrt{y})+\exp(-2\sqrt{y})\} \left\{ \frac{b}{c}(1-x)^{c-a-b} + \left(1-\frac{b}{c}\right)(1-x)^{-b} \right\} \quad \ldots(6.3)$$

where $\Xi_2$ is defined by $\Xi_2(a;b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_m x^m y^n}{(c)_{m+n}\ m!\ n!}$, $|\ x\ | < 1$, $\quad \ldots(6.4)$

$c$ is not a negative integer or zero.

To prove Theorem 10, express (6.2) as

$$\Phi_3(b;c;x,y) = \sum_{m=0}^{\infty} \frac{(b)_m x^m}{(c)_m\ m!}\ {}_0F_1\ (-;\ c+m;y), \quad \ldots(6.5)$$

and apply Kummer's second formula (Rainville 1971, Th. 43, p. 126). Repeated application of (2.13) then yields the desired theorem. The proof of Theorem 11 is similar to that of Theorem 10.

Although, both theorems hold for an arbitrary positive $y$, these inequalities are sharp only for $y$ in the neighbourhood of zero.

In particular, theorem 10 gives

$$1.22140276 < \Phi_3\ (.5;\ 1;\ .4;\ .01) \leqslant 1.27091377, \quad \ldots(6.6)$$

and Theorem 11 gives

$$1.4150117 < \Xi_2\ (3;\ .5;\ 1;\ .8,\ .1) < 1.593736. \quad \ldots(6.7)$$

## 7. Inequalities for $\Psi_2$

The foregoing techniques cannot successfully be applied in the case of $\Psi_2$, which is defined by

$$\Psi_2 \equiv \Psi_2(a;\ c,\ c'\ ;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\ x^m y^n}{(c)_m (c')_n m!n!} \quad \ldots(7.1)$$

$c$ and $c'$ are not negative integers or zero.

An inequality, though not very simple and sharp, may however be given using improved version of Buschman's theorem (1976, Th. 2) as given by us (Joshi and Arya 1981, Th. 4)

$$\left.\begin{aligned} &g(x)L < {}_1F_1(a+n+1;\ c;\ x) < g(x)U,\\ &\text{where } g(x)=(x-c+2(a+n))/(a)_{n+1},\\ &U=({}_1F_1(a;c;x)+{}_1F_1(\mid a-1\mid;c;x))3^n((x-c)/3)+a)_n,\\ &L=({}_1F_1(a;c;x)-{}_1F_1(\mid a-1\mid;c;x))(x-c+a)_n,\end{aligned}\right\} \qquad \ldots(7.2)$$

provided $a \geqslant 1/2$, $0<c \leqslant a$, or $x > \max\{c, 2(c-a)\} > 0$, and for $a>1$, absolute value symbol in ${}_1F_1$'s can be dropped. Expressing (7.1) in the form

$$\Psi_2 \equiv \Psi_2(a;c,c';x,y) \sum_{m=0}^{\infty} \frac{(a)_m x^m}{(c)_m m!}\ {}_1F_1(a+m;c';y), \qquad \ldots(7.3)$$

and applying (7.2) and approximating the resulting ${}_1F_1$'s by (2.13), we have the following theorem:

*Theorem* 12—If (i) $a>2$ (ii) $c>y-c'+a-1>0$, $x>0$, (iii) $c'+2>a>c'+1$, $y>0$ or $y>\max\{c', 2(c'-a)\}>0$, $c'+1>a$, then

$$\frac{y}{c'} e^{(a-1)y/(1+c')} \left\{ \frac{y-c'+2(a-1)}{(a-1)} e^{(y-c'+a-1)/y} + \frac{2x(y-c'+a-1)}{c(a-1)} e^{x(y-c'+a)/(1+c)} \right\}$$

$$< \Psi_2 < \left(2 + \frac{(2a-3)(e^y-1)}{c'}\right) \left\{ \frac{y-c'+2(a-1)}{(a-1)} \left[1 + \frac{(y-c'+3(a-1))(e^{3x}-1)}{3c}\right]\right.$$

$$\left. + \frac{2x(y-c'+3(a-1))}{c(a-1)} \left[1 + \frac{(y-c'+3a)(e^{3x}-1)}{3(1+c)}\right]\right\}. \qquad \ldots(7.4)$$

In conclusion it is noted that because of certain basic difficulties, the discussion on the inequalities either of $G_1, \ldots, H_7$ in the Horn's List or the remaining confluent functions has been postponed for a future investigation.

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