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SOME EXACT INTERIOR SOLUTIONS OF NONSTATIC AXISYMMETRIC SPACE-TIME EINSTEIN'S FIELD EQUATIONS

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Relativistic field equations for axisymmetric gravitational collapse with escaping neutrinos are studied. Interior solutions of Einstein's field equations for an oblate spheroid of perfect fluid, which emites neutrinos, are discussed. It is shown that the fluid is in rigid rotation. A class of exact nonstatic, interior solutions representing gravitational field of a fluid spheroid with positive density and pressure, are presented. On the boundary of an oblate spheroidal source, the metric of the interior space matches with the metric of enveloping radiation zone.

1. INTRODUCTION

Relativistic field equations for the gravitational collapse with escaping neutrinos of a perfect fluid sphere was presented by Misner (1965). In obtaining relativistic field equations, he assumes that at extreme temperature, the collapsing perfect fluid sphere emites neutrinos in radial direction, which have no subsequent interaction with matter.

Quasi stellar radio sources are rotating objects. Due to rotation a self gravitating collapsing source becomes axially symmetric. We (Patel 1979) have presented relativistic field equations for axisymmetric gravitational collapse with escaping neutrinos. In deriving these equations it is assumed that the escaping neutrinos revolve around the source and they are neither scattered nor absorbed by the surrounding matter. Since each element of the fluid cools by emission of neutrinos at some rate depending on temperature and density, the fluid does not obey the simple adiabatic equation of state. Here in this paper, exact solutions of Einstein's field equations for the dynamics of a gravitational collapse with escaping neutrinos of an oblate spheroidal fluid source are obtained. It is proved that the collapsing fluid is in rigid rotation and energy density and pressure are positive. Also a class of exact solutions for a special case of axisymmetric gravitational collapse without neutrinos emission is obtained.

2. THE STRESS-ENERGY TENSOR

The matter conservation is expressed by the equation of continuity

$$(N\,u^j)_{;j} = 0 \qquad \ldots(2.1)$$

where N is baryon number density and u^i is a fluid four-velocity vector. The stress-energy tensor

$$T^{ij} = (\rho + p)\, u^i u^j + p\, g^{ij} \qquad \ldots(2.2)$$

fails to satisfy a local conservation law because of neutrinos emission. If $C\,(T, N)$ is a colling rate of decrease in internal energy due to neutrinos emission per unit amount of matter, then $\quad -NC = u^i\,(-T_i{}^j{}_{;j}).$ $\qquad \ldots(2.3)$

Let us define $\rho = N\,(1 + e)$ $\qquad \ldots(2.4)$

where e is specific internal energy that does not include rest mass-energy. Now using eqns. (2.2) and (2.3) we get

$$e_{,i}\, u^i = -\,C - \frac{p}{N}\; u^i{}_{;i}\,. \qquad \ldots(2.5)$$

This is the first law of thermodynamics with $-C$, as the heat input rate. Here and in the following comma and semicolon signifies partial and covariant differentiation respectively, with respect to the index that follow.

The neutrinos flux is defined by the stress-energy tensor

$$E^{ij} = q\, k^i\, k^j\,; \qquad k^i\, k_i = 0 \qquad \ldots(2.6)$$

where q is density of energy flux in some frame which depends on normalization of k^i. The local energy momentum conservation law is

$$(T^{ij} + E^{ij})_{;j} = 0. \qquad \ldots(2.7)$$

Use of eqn. (2.3) simplifies the conservation law (2.7) to the form

$$u^i\,(-E^j_{i\,;j}) = NC\,. \qquad \ldots(2.8)$$

This equation represents the behaviour of the neutrinos flux.

3. Metric and Field Equations

Oblate spheroidal co-ordinates system is a suitable co-ordinates system for study of the gravitational collapse of a rotating star. As discussed by the author (Patel 1978 a,b) axially symmetric space-time line element, in an oblate spheroidal co-ordinates system, is of the form

$$ds^2 = -\,e^{2\psi}\, dt^2 + e^{2\sigma}\,(d\phi - w\, dt)^2 + e^{2\beta}\, a^2\,(\theta^2 + \alpha^2)\left[\frac{d\theta^2}{1+\theta^2} + \frac{d\alpha^2}{1-\alpha^2}\right] \qquad \ldots(3.1)$$

where ψ, σ, w and β are functions of t, θ and α, and $0 \leqslant \theta < \infty\,; -1 \leqslant \alpha \leqslant 1$.

Let us make some general remarks on the notations. The co-ordinates are numbered as $x^0 = t$, $x^1 = \phi$, $x^2 = \theta$ and $x^3 = \alpha$. Latin letter as a space-time index is allowed the range 0, 1, 2 and 3 while Greek letter is restricted to 2 and 3 only. Summation is implied by repeated indices and restricted to their ranges.

An axially symmetric space-time restricts the motion along the directions of x^2- and x^3-axes, only rotational motion in the ϕ-direction, specified by Ω can prevail. As discussed by Chandrasekhar and Friedman (1972), the components of the four-velocity vector are

$$u^0 = \frac{e^{-\psi}}{(1-V^2)^{1/2}} \; ; u^1 = \frac{we^{-\psi}+Ve^{-\sigma}}{(1-V^2)^{1/2}} \; ; u^2 = u^3 = 0, \qquad ...(3.2)$$

where $V = (\Omega - w)\, e^{(\sigma-\psi)}$. ...(3.3)

The emitted neutrinos form an enveloping radiation zone surrounding a rotating axially symmetric collapsing star. And according to our assumption the neutrinos are neither scattered nor absorbed by the surrounding matter. Therefore, as discussed by author (Patel 1978b) the null vector k^i takes the normal form

$$k^i = (e^{-\psi}, \, w\, e^{-\psi} - e^{-\sigma}, \, 0 \, , 0). \qquad ...(3.4)$$

Now, use of eqns. (2.2) and (2.3) simplify the local energy momentum conservation law (2.7) into the form

$$(\rho + p)\, u^i{}_{;j}\, u^j = NC\, u^i - (g^{ij} + u^i u^j) p_{,j} - E^{ij}{}_{;j}. \qquad ...(3.5)$$

Equation (2.8), which represents the behaviour of the neutrinos flux reduces to

$$q_{,0} + 2q\,(\sigma + \beta)_{,0} = NC\, e^{\psi}\left\{\frac{1-V}{1+V}\right\}^{1/2}. \qquad ...(3.6)$$

Then $t-$ and $\phi-$components of eqn. (3.5) simplify to a form

$$(\rho + p)\,\{V_{,0} + V(1-V^2)\,\sigma_{,0}\}\{1-V^2\}^{-1}$$
$$= NC\, e^{\psi}\,(1-V^2)^{1/2} - V\, p_{,0}. \qquad ...(3.7)$$

The equations representing conservation of momentum conjugate to $\theta-$ and $\alpha-$directions are

$$\frac{(\rho+p)}{1-V^2}\left(\Psi_{,\lambda} - V^2\sigma_{,\lambda} + V\, w_{,\lambda}\, e^{(\sigma-\psi)} \right)$$
$$= -p_{,\lambda} - q\,\{(\Psi-\sigma)_{,\lambda} - w_{,\lambda}\, e^{(\sigma-\psi)}\}. \qquad ...(3.8)$$

The following are Einstein's field equations

$$w_{,0\lambda} + w_{,\lambda}\,(3\sigma - \psi)_{,0} = 0, \qquad ...(3.9)$$

$$(\sigma + \beta)_{,0\lambda} + (\sigma - \beta)_{,0}\, \sigma_{,\lambda} - (\sigma + \beta)_{,0}\, \psi_{,\lambda} = 0, \qquad ...(3.10)$$

$$-(\theta^2 + \alpha^2)\, e^{-2\psi}\,(\beta_{,00} + \beta_{,0}\,(\sigma - \psi + 2\beta)_{,0})$$
$$+ \frac{e^{-2\beta}}{a^2(\theta^2+\alpha^2)}\,(X - Y + (2 + \theta^2 - \alpha^2)\{ (1-\alpha^2)\,\beta_{,3}\,(\sigma + \psi)_{,3}$$
$$-(1+\theta^2)\,\beta_{,2}\,(\sigma + \psi)_{,2} \}$$
$$+ (\theta^2 + \alpha^2)\,\Delta\,\beta + \frac{2(1+\theta^2)(1-\alpha^2)}{\theta^2+\alpha^2}\,\{\alpha(\sigma+\psi)_{,3} - \theta\,(\sigma+\psi)_{,2}\})$$
$$- \frac{e^2(\sigma-\psi-\beta)}{2a^2(\theta^2+\alpha^2)}\left((1 + \theta^2)^2\,(w_{,2})^2 - (1-\alpha^2)^2\,(w_{,3})^2 \right) = 0, \qquad ...(3.11)$$

$$(\sigma + \psi)_{,23} - \left(\beta_{,3} + \frac{\alpha}{\theta^2+\alpha^2} \right)(\sigma + \psi)_{,2} - \left(\beta_{,2} + \frac{\theta}{\theta^2+\alpha^2} \right)(\sigma + \psi)_{,3}$$
$$+ \psi_{,2}\,\psi_{,3} + \sigma_{,2}\,\sigma_{,3} - \tfrac{1}{2}\, w_{,2}\, w_{,3}\, e^{2(\sigma-\psi)} = 0, \qquad ...(3.12)$$

$$e^{-2\psi}\,[\tfrac{1}{2}\,(\theta^2 - \alpha^2)\,\{\beta_{,00} + \beta_{,0}\,(\sigma - \psi + 2\beta)_{,0}\} + \beta_{,0}\,(2\sigma + \beta)_{,0}]$$
$$+ \frac{e^{-2\beta}}{2a^2(\theta^2+\alpha^2)}\,[\,(\theta^2 + \alpha^2)\,\{ (1 + \theta^2)\,\beta_{,2}\,(\sigma + \psi)_{,2}$$
$$-(1-\alpha^2)\,\beta_{,3}\,(\sigma + \psi)_{,3} \} + \Delta\,(\psi - \sigma) - X - Y - (2+\theta^2-\alpha^2)\Delta\beta\,]$$
$$- \frac{e^2(\sigma-\psi-\beta)}{4a^2(\theta^2+\alpha^2)}\left[2w\,Z + (1-\theta^4)\,(w_{,2})^2 + (1-\alpha^4)\,(w_{,3})^2 \right]$$
$$= -8\pi\,[p - (1-V^2)^{-1}\,(p + \rho)\,\{ 1 + Vwe^{(\sigma-\psi)} \} - q\,\{ 1 - we^{(\sigma-\psi)} \}\,], \qquad ...(3.13)$$

$$e^{-2\psi}\,[\,(\beta+\sigma)_{,00}+\beta_{,0}\,(\sigma-\psi+\beta)_{,0}+\sigma_{,0}\,(\sigma-\psi)_{,0}\,]$$
$$-\frac{e^{-2\beta}}{a\,(\theta^2+\alpha^2)}\,[\,\tfrac{1}{2}\,\Delta\,(\sigma+\psi)+(1+\theta^2)\,\psi_{,2}\,(\sigma+\psi)_{,2}$$
$$+(1-\alpha^2)\,\psi_{,3}\,(\sigma+\psi)_{,3}\,]=-8\pi p, \qquad \ldots(3.14)$$
$$e^{-2\psi}\,[\,\tfrac{1}{2}\,(\theta^2-\alpha^2)\,\{\beta_{,00}-\beta_{,0}\,(\sigma-\psi+2\beta)_{,0}\}+2\beta_{,00}-\beta_{,0}\,(2\,\psi-3\beta)_{,0}\,]$$
$$+\frac{e^{-2\beta}}{2a^2(\theta^2+\alpha^2)}\,[\,(\theta^2+\alpha^2)\,\{\,(1+\theta^2)\,\beta_{,2}\,(\sigma+\psi)_{,2}$$
$$-(1-\alpha^2)\,\beta_{,3}\,(\sigma+\psi)_{,3}\}+\Delta\,(\sigma-\psi)-X-Y$$
$$-(2+\theta^2-\alpha^2)\,\Delta\,\beta+2\,(1+\theta^2)\,(\sigma+\psi)_{,2}\,(\sigma-\psi)_{,2}$$
$$+2(1-\alpha^2)\,(\sigma+\psi)_{,3}\,(\sigma-\psi)_{,3}\,]$$
$$+\frac{e^{2(\sigma-\psi-\beta)}}{4a^2(\theta^2+\alpha^2)}\,[\,2w\,Z+(1+\theta^2)\,(3+\theta^2)\,(w_{,2})^2+(1-\alpha^2)\,(3-\alpha^2)\,(w\,e_{,3})^2\,]$$
$$=-8\pi\,[\,(1-V^2)^{-1}\,\{p\,(1+V\,w\,e^{(\sigma+\psi)})+\rho\,V\,(V+w\,e^{(\sigma-\psi)})\,\}$$
$$+q\,\{1-w\,e^{(\sigma-\psi)}\}\,] \qquad \ldots(3.15)$$
$$e^{(\sigma-\psi-2\beta)}\,Z=16\,\pi\,a^2\,(\theta^2+\alpha^2)\,[\,q-(1-V^2)^{-1}\,V\,(\rho+p)\,] \qquad \ldots(3.16)$$
$$2\,w\,e^{-2\psi}\,[-\beta_{,00}+\beta_{,0}\,(\sigma+\psi-\beta)_{,0}\,]$$
$$+\frac{e^{-2\beta}}{2a^2(\theta^2+\alpha^2)}\,[\,-Z-2w\,\{\Delta\,(\sigma-\psi)-(1+\theta^2)\,(\psi+\sigma)_{,2}\,(\psi-\sigma)_{,2}$$
$$-(1-\alpha^2)\,(\psi+\sigma)_{,3}\,(\psi-\sigma)_{,3}\}\,]$$
$$-\frac{e^{2(\sigma-\psi-\beta)}}{2a^2(\theta^2+\alpha^2)}\,[\,2w\,\{(1+\theta^2)\,(w_{,2})^2+(1-\alpha^2)\,(w_{,3})^2\}+w^2\,z\,]$$
$$=8\,\pi e^{\psi-\sigma}\left[\frac{\rho+p}{1-V^2}\,\{1+V\,we^{(\sigma-\psi)}\}\,\{V+w\,e^{(\sigma-\psi)}\}-q\,\{1-we^{(\sigma-\psi)}\,\}^2\,\right]. \qquad \ldots(3.17)$$

where $$\Delta\equiv(1+\theta^2)\,\frac{\partial^2}{\partial\theta^2}+(1-\alpha^2)\,\frac{\partial^2}{\partial\alpha^2}+\theta\,\frac{\partial}{\partial\theta}-\alpha\,\frac{\partial}{\partial\alpha} \qquad \ldots(3.18)$$
$$X\equiv(1+\theta^2)^2\,[\,(\sigma+\psi)_{,22}+(\psi_{,2})^2+(\sigma_{,2})^2\,] \qquad \ldots(3.19)$$
$$Y\equiv(1-\alpha^2)^2\,[\,(\sigma+\psi)_{,33}+(\psi_{,3})^2+(\sigma_{,3})^2\,] \qquad \ldots(3.20)$$
$$Z\equiv\Delta\,w+(1+\theta^2)\,w_{,2}\,(3\,\sigma-\psi)_{,2}+(1-\alpha^2)\,w_{,3}\,(3\,\sigma-\psi)_{,3}\,, \qquad \ldots(3.21)$$

4. Solution for Interior Space

It has been proved that in a region of an axially symmetric enveloping radiation zone (see Patel M. D. (1978b))

$$\sigma+\psi=\text{Constant}. \qquad \ldots(4.1)$$

And according to our assumption, the rotating source remains axisymmetric. Therefore, on the oblate spheroidal boundary of the source, the interior metric should match to the metric of an axially symmetric enveloping radiation zone. Hence for the interior space of the source one can modify the condition (4.1) into the form

$$\sigma+\psi=\log U, \qquad \ldots(4.2)$$

where U is a function of t only.

A first integral of equation (3.9) is $w_{,\lambda}=A\,e^{(\psi-3\sigma)}=A\,U\,e^{-4\sigma}$, where A is an arbitrary function of $\theta+\alpha$. Then equation (3.12) reduces to

$$\frac{\partial}{\partial\theta}\,(e^{2\sigma})\,\frac{\partial}{\partial\alpha}\,(e^{2\sigma})=A^2. \qquad \ldots(4.3)$$

Let us define

$$\sigma = \frac{1}{4} f(\theta + \alpha). \qquad ...(4.4)$$

Then eqn. (4.2) gives

$$\psi = \log U - \tfrac{1}{4} f. \qquad ...(4.5)$$

Using the results (4.4) and (4.5) in eqn. (3.9), we get

$$w = U e^{-f/2}. \qquad ...(4.6)$$

Now, eqn. (3.10) simplifies to the form

$$\beta_{,0\lambda} = 0, \qquad ...(4.7)$$

which implies a simple result

$$\beta = G(\theta, \alpha) + S(t) \qquad ...(4.8)$$

where G is an arbitrary function of θ and α while S is an arbitrary function of t only. Use eqns. (4.4) to (4.6) and (4.8) in eqn. (3.11) and simplify the result. Then the resulting equation split into two equations

$$e^{2S} \left[\ddot{S} + 2(\dot{S})^2 - \frac{\dot{S}\dot{U}}{U} \right] = m U^2,$$

$$\Delta G = m a^2 (\theta^2 + \alpha^2) e^{(2G+f/2)}$$

where m is a constant parameter and overhead dot represents differentiation with respect to t. It has been proved by the author (Patel 1978 b) that in a region of enveloping radiation zone $\Delta G = 0$. Therefore on the oblate spheroidal boundary of the source also $\Delta G = 0$, which implies $m = 0$. Hence we get two equations

$$e^{2S}\left[\ddot{S} + 2(\dot{S})^2 - \frac{\dot{S}\dot{U}}{U} \right] = 0 \qquad ...(4.9)$$

$$\Delta G = 0. \qquad ...(4.10)$$

Now, the local energy conservation equations (3.6) to (3.8) reduce to the forms

$$q_{,0} + 2q\dot{S} = NCUe^{-f/4} \left\{ \frac{1-V}{1+V} \right\}^{1/2} \qquad ...(4.11)$$

$$\frac{(\rho+p)V^2_{,0}}{(1-V^2)^{3/2}} - \frac{2V^2 p_{,0}}{(1-V^2)^{1/2}} = 2NCUVe^{-f/4} \qquad ...(4.12)$$

$$\frac{\rho+p}{4} \left\{ \frac{1+V}{1-V} \right\} f_{,\lambda} = p_{,\lambda}. \qquad ...(4.13)$$

Also Einstein's field equations (3.13) to (3.15) and (3.17) simplify to

$$\rho + pV = p + \rho V = (1-V)p = \tfrac{1}{2}(\rho+p)(1+V) = \frac{1-V}{8\pi} e^{f/2} \left\{ \frac{\dot{S}}{U} \right\}^2. \qquad ...(4.14)$$

In general these equations are not consistence. They are consistence only if $V=0$. (i.e. $\Omega = w$). Physically it means that the fluid source is in rigid rotation. Equation (3.16), which prescribes energy density of radiation, reduces to

$$q = -\frac{e^{-2(G+S+f/4)}}{16\pi a^2(\theta^2+\alpha^2)} \Delta(e^{f/2}) \qquad ...(4.15)$$

General solution of eqn. (4.9) is

$$e^{2S} = bR + d \qquad ...(4.16)$$

where b and d are arbitrary constants and R is function of t defined as $\dot{R}=U$. Now, the equations prescribing pressure and energy density of the fluid take a simple form

$$p=\rho=\frac{1}{32\pi}e^{f/2}\left\{\frac{b}{bR+d}\right\}^2. \qquad \ldots(4.17)$$

General solution of eqn. (4.10) can be obtain by using the method of separation of variables and Frobenious method (1956). The complete integral of eqn. (4.10) is

$$G=\Big(C_0\Big\{1+\sum_{s=1}^{\infty}\frac{m^2(m^2-2^2)\ldots\{m^2-(2s-2)^2\}}{(2s)\,!}\theta^{2s}\Big\}$$
$$+C_1\theta\Big\{1+\sum_{s=1}^{\infty}\frac{(m^2-1)(m^2-3^2)\ldots\{m^2-(2s-1)^2\}}{(2s+1)\,!}\theta^{2s}\Big\}\Big)$$
$$\times\Big[D_0\Big\{1+\sum_{s=1}^{\infty}\frac{(-m^2)(2^2-m^2)\ldots\{(2s-2)^2-m^2\}}{(2s)\,!}\alpha^{2s}\Big\}$$
$$+D_1\alpha\Big\{1+\sum_{s=1}^{\infty}\frac{(1^2-m^2)(3^2-m^2)\ldots\{(2s-1)^2-m^2\}}{(2s+1)\,!}\alpha^{2s}\Big\}\Big] \qquad \ldots(4.18)$$

where C_0, C_1, D_0 and D_1 are arbitrary constants and m is a constant parameter of the family.

Now let us consider a special case of axially symmetric source which is not emitting neutrions. In this case $q=0$ and hence

$$\Delta(e^{f/2})=0. \qquad \ldots(4.19)$$

The solution of (4.19) is

$$e^{f/2}=\Big[C_0'\Big\{1+\sum_{s=1}^{\infty}\frac{n^2(n^2-2^2)\ldots\{n^2-(2s-2)^2\}}{(2s)\,!}\theta^{2s}\Big\}$$
$$+C_1'\theta\Big\{1+\sum_{s=1}^{\infty}\frac{(n^2-1^2)(n^2-3^2)\ldots\{n^2-(2s-1)^2\}}{(2s+1)\,!}\theta^{2s}\Big\}\Big]$$
$$\times\Big[D_0'\Big\{1+\sum_{s=1}^{\infty}\frac{(-n^2)(2^2-n^2)\ldots\{(2s-2)^2-n^2\}}{(2s)\,!}\alpha^{2s}\Big\}$$
$$+D_1'\alpha\Big\{1+\sum_{s=1}^{\infty}\frac{(1^2-n^2)(3^2-n^2)\ldots\{(2s-1)^2-n^2\}}{(2s+1)\,!}\alpha^{2s}\Big\}\Big] \qquad \ldots(4.20)$$

where C_0', C_1', D_0' and D_1' are arbitrary constants and n is a parameter of the family.

If we select the family parameter $n=1$, then above solution reduces to a simple form

$$e^{f/2}=\left\{ C_0'(1+\theta^2)^{1/2}+C_1'\theta \right\}\left\{ D_0'(1-\alpha^2)^{1/2}+D_1'\alpha \right\}. \quad ...(4.21)$$

Since $f^{f/2}>0$, C_0', C_1' and D_0' are positive constants and

$D_1'=0$. Therefore

$$e^{f/2}=\{C_1(1+\theta^2)^{1/2}+C_2\theta\}\{(1-\alpha^2)^{1/2}\}, \quad ...(4.22)$$

where C_1 and C_2 are positive constants. Now the equation giving pressure and density is

$$p=\rho=\frac{1}{32\pi}\left\{\frac{b}{bR+d}\right\}^2\left\{C_1(1+\theta^2)^{1/2}+C_2\theta\right\}\left\{1-\alpha^2\right\}^{1/2}. \quad ...(4.23)$$

5. Conclusion

On the boundary of an oblate spheroidal source, the metric of the interior space-time matches with the metric of enveloping radiation zone. Both the pressure and density of the fluid are positive and equal. In a special case of a source, not emitting neutrinos, the gravitational force is counterbalanced by the forces due to internal pressure and rotation.

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