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THE APPLICATION OF NEWTON'S METHOD TO THE SOLUTION OF A FREE-BOUNDARY PROBLEM ARISING IN THE OIL INDUSTRY

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The problem is formulated in terms of a flow potential function which satisfies Laplace's equation in axial symmetry. This is solved using a finite-difference scheme with a variable mesh spacing tailored to suit the curved part of the boundary (the free-boundary) whose shape has to be determined. Standard conditions are imposed on the fixed part of the boundary, but on the free part both the potential function and its normal derivative are prescribed. The method is iterative, and at each iteration the Laplace equation is solved with the current boundary shape by satisfying one of the free boundary conditions, the shape then being improved to approximate the second condition more nearly using Newton's method independently for each mesh point of the boundary.

1. INTRODUCTION

Most comm ercial oil reservoirs consist of a layer of sand containing oil and water bounded above by an impermeable layer. Since the density of oil is less than that of water the oil lies above the water in equilibrium, the interface between the two being horizontal. An oil well consists essentially of a cylindrical hole through the impermeable layer and some distance, d say, into the oil-bearing sand. As oil is extracted through the well; assuming homogeneity in all horizontal layers there will be symmetry about the axis of the well. If oil is extracted at a constant rate the oil-surface interface assumes an equilibrium position typically as shown in Fig. 1. (referred as coning). The oil producer wishes to extract oil as quickly as possible, but not so quickly that the oil-water interface reaches the bottom of well, for then the oil would be polluted by water.

The physical problem can be modelled mathematically using velocity potential, ϕ, which must satisfy Laplace's equation with the standard boundary conditions on the fixed boundaries, but with two conditions on the free boundary, that is, the oil-water interface.

The classical paper by Muskat and Wyckoff (1935) was an extension of an earlier paper by Muskat (1932) in which an approximate analytic solution was obtained under the assumption that the velocity potential in the oil with a cone-shaped lower

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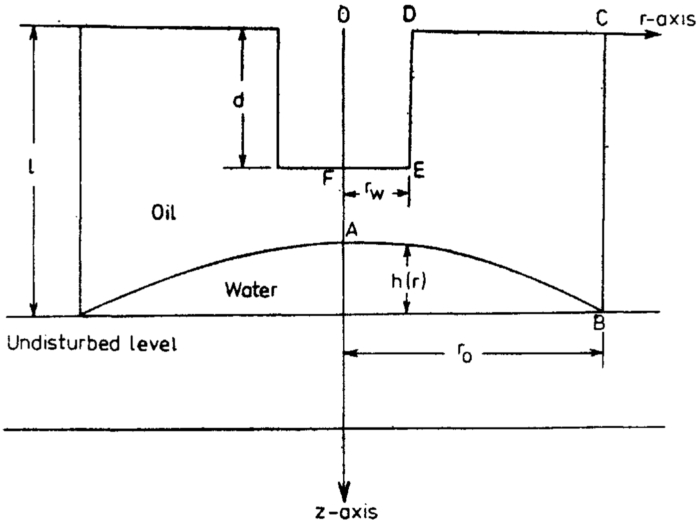

FIG. 1

interface were plane. Such an assumption, as Muskat points out, makes his results qualitative only.

This and related problems have been solved approximately by Arthur (1944), Meyer and Gardner (1954), Karp *et al* (1962), Smith and Pirson (1963) and Welge and Weber (1964). Karplus (1956) used an electronic analogue to obtain a solution of the oil-water coning problem.

In the present paper an iterative finite-difference method of solution of the water-coning problem is given. The derivation of the mathematical model is given in section 2, the finite-difference system is obtained in section 3, and computational details, numerical results and comparisons are given in sections 4 and 5 respectively.

2. Derivation of Model

Various idealizations have to be made in the physical system to obtain a reasonable mathematical model. The oil is assumed to be incompressible and homogeneous, the porous medium (sand) isotropic and homogeneous and the flow isothermal. The boundary-layer at the oil-water interface is assumed to be of zero thickness and the oil-water layers to extend to infinity in all directions.

The vector velocity in the oil, v, may be written as

$$\mathbf{v} = -\nabla\phi,$$

where ϕ is the velocity potential. This satisfies Laplace's equation which in cylindrical polar coordinates with axial symmetry is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}=0. \qquad \ldots(2.1)$$

The following boundary conditions (see Fig. 1) are assumed:

$$\phi = \text{constant} = \phi_w \qquad \ldots(2.2)$$

over the surface of the well,

$$\phi = \text{constant} = \phi_e \qquad \ldots(2.3)$$

at an assumed drainage radius r_e.

$$\frac{\partial \phi}{\partial z} = 0 \text{ on } z=0,\ r \geqslant r_w' \qquad \ldots(2.4)$$

where r_w is the radius of the well.

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } z = h(r),\ r \geqslant 0 \qquad \ldots(2.5)$$

where $h(r)$ is the height of the oil-water interface above its undisturbed level, n is the outward normal to the surface.

Since the oil-water interface position is to be determined, another condition is needed on it, and this is provided by an empirical law due to Darcy given in Bear (1972). This states that on $z=h(r)$

$$\phi = \frac{k}{\mu} (g\rho_0 z - p)$$

where p is the excess of the hydrostatic pressure over that at $z=0$, z is the depth below the upper surface of the oil, k is the permeability of the sand and μ and ρ_0 are the viscosity and density of the oil respectively.

Since $p = \rho_0 g l - \rho_w g h$,

where ρ_w is the density of water, and l is the depth of the undisturbed oil-layer, we obtain

$$\phi = \frac{k}{\mu}\left\{g\rho_0(l-h) - \rho_0 g l + \rho_w g h\right\} = g\frac{k}{\mu}\left(\rho_w - \rho_0\right) h = ch(r), \text{ say.} \qquad \ldots(2.6)$$

3. Numerical Method

An iterative method is used to generate a sequence of approximations $c^{(k)}$ to the free boundary AB and the velocity potential $\phi^{(k)}$ satisfying eqns. (2.1)–(2.4) and eqn. (2.5) on $C^{(k)}$. $\phi^{(k)}$ will generally not satisfy eqn. (2.6) and the boundary is thus moved to $C^{(k+1)}$ to better approximate eqn. (2.6) using Newton's method.

The method of computing $\phi^{(k)}$ is given in section 3.1 and that for $C^{(k)}$ in section 3.2

3.1. *Finite-difference Solution for* $\phi^{(k)}$

A non-uniform rectangular mesh is used; the spacing in the r-direction is chosen arbitrarily out to a large finite radius $r=r_e$ where ϕ is made zero. The spacing is normally closer for smaller r, particularly near the well, to represent the gradients in ϕ better. The spacing in the z-direction is again chosen arbitrarily except that the bottom of the well corresponds to a grid line and the spacing for that part of the mesh below the maximum height of the oil-water interface is chosen so that mesh corners lie on the oil-water interface as shown in Fig. 2. This interface is effectively represented by the polygon formed by the diagonals of meshes lying on it.

We take the vertical mesh lines as $r=r_j$, $j=1,2,\ldots, N$; with spacing $s_j=r_j-r_{j-1}$ and horizontal lines as $z=z_i$, $i=1,2,\ldots,M$; with $p_i=z_i-z_{i-1}$. Following Varga (1962),

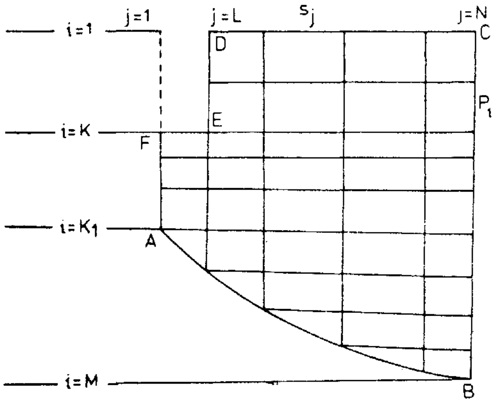

FIG. 2

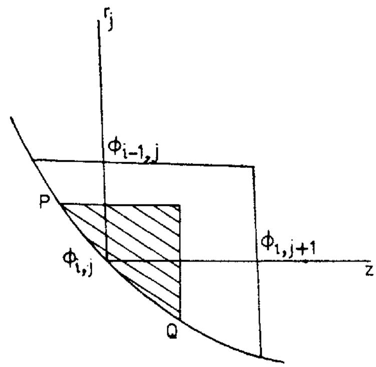

FIG. 3

we associate with each point (z_i, r_j) interior to the grid an area R_{ij} bounded by the lines, $z=z_i-\frac{1}{2}p_i$, $z=z_i+\frac{1}{2}p_{i+1}$, $r=r_j-\frac{1}{2}s_j$ and $r=r_j+\frac{1}{2}s_{j+1}$.

For points on the oil-water interface R_{ij} is a quadrilateral as shown in Fig. 3.

For each gridpoint (z_i, r_j) where the potential is unknown, Green's theorem is applied to the region R_{ij} to give

$$\iint_{R_{ij}} \nabla^2\phi dr\, dz = \int_{C_{ij}} r\left(\frac{\partial\phi}{\partial r}dz - \frac{\partial\phi}{\partial z}dr\right) \quad \text{.. (3.1)}$$

where C_{ij} is the boundary of R_{ij}.

At internal points the line integral in (3.1) is approximated by central differences, giving

$$E_{ij}\phi_{ij}-A_{ij}\phi_{i+1,j}-B_{ij}\phi_{i-1,j}-C_{ij}\phi_{i,j+1}-D_{ij}\phi_{i,j-1}=0 \quad \text{.. (3.2)}$$

where $A_{ij} = r_j(s_j+s_{j+1})/p_{i+1}$, $B_{ij} = r_j(s_j+s_{j+1})/p_i$, $C_{ij} = r_{j+1/2}(p_i+p_{i+1})s_{j+1}$,

$D_{ij} = r_{j-1/2}(p_i+p_{i+1})/s_j$, $E_{ij} = A_{ij}+B_{ij}+C_{ij}+D_{ij}$

and $r_{j+\frac{1}{2}}=r_j+\frac{1}{2}s_{j+1}$, $r_{j-1/2} = r_{j-1/2}s_j$.

For points along the free boundary AB, the line integral takes the simpler form

$$\tfrac{1}{2}(p_i+p_{i+1})\{r_{j+1/2}(\phi_{ij}-\phi_{i,j+1})/s_{j+1}\}+\tfrac{1}{2}(s_js_{j+1})$$
$$\{r_j(\phi_{ij}-\phi_{i-1,j})/p_i\}+\int_p^Q r\partial\phi/\partial n\, ds=0.$$

The boundary condition (2.5) gives $\partial\phi/\partial n=0$ on AB, and so the finite difference equations for the points on the boundary may be written as

$$F_{ij}\phi_{ij}-B_{ij}\phi_{i-1,j}-C_{ij}\phi_{i,j+1}=0$$

where $F_{ij}=E_{ij}-A_{ij}-D_{ij}$.

Expressions can be obtained similarly for mesh points (z_i, r_j) on the other boundaries CD and AF, where ϕ_{ij} is unknown. The resulting finite difference equations are solved by using successive over-relaxation.

3.2. *Moving the Oil-water Interface*

The boundary grid-points are moved individually along lines $r = r_j$ using Newton's method applied to the function

$$f(z_i) = \phi(z_i,r_j) - ch(r_j)$$

since the solution we require to satisfy equation (2.6) is $f(z_i)=0$. With $h(r_j) = l-z_j$, the equations for the new z values are

$$z_i^{(k+1)} = z_i^k - \frac{\phi(z_i^{(k)}, r_j) - C(l - z_i^{(k)})}{\phi_z(z_i^{(k)}, r_j) + C}$$

$\phi(z_i^{(k)}, r_j)$ is known already and $\phi_z(z_i^{(k)}, r_j)$ is calculated from the three-point formula

$$\phi_z(z_i^{(k)}, r_j) = \left\{(2p_i+p_{i-1})/p_i(p_i+p_{i+1})\right\}\phi(z_i^{(k)}, r_j)$$
$$-\left\{(p_i+p_{i-1})/p_ip_{i-1}\right\}\phi(z_{i-1}^{(k)}, r_j)+\left\{(p_i/p_{i-1})(p_i+p_{i-1})\right\}\phi(z_{i-2}^{(k)}, r_j).$$

After calculating the new boundary points the mesh spacing in the z-direction is automatically adjusted to put the corners of the mesh on the boundary again.

4. Computational Details

A starting position for the oil-water interface $C^{(0)}$ provided by the curve

$$z = (A+Br)^{1/2}$$

with A and B chosen to make $z=l$ when $r=r_e$, and to make z at $r=0$ equal to the value obtained from earlier results by Muskat and Wyckoff (1935).

Several sizes of mesh were used to make sure that the solution was converging correctly. For the finer mesh cases r_e was halved, values for ϕ on this boundary being taken from the coarser calculations.

Values of the over-relaxation parameter of between 1.8 and 1.95 were found most effective in keeping down the number of iterations for solving the Laplace equation. Convergence was assumed when the maximum difference between the components of two successive iterates was less than 10^{-6}. This number roughly decreased from 1000 for $k=0$, to 16 for $k=20$, when the problem was assumed to have converged when successive values of $z_j^{(k)}$ differed by less than 10^{-5} each j.

5. Results and Comparisons

Two cases have been calculated using parameter values $d=24$, $\phi_e = 0$, $l = 48$, $r_w=1$, $r_e=96$ and $c=0.35$, one with $\phi_w=10$ and one with $\phi_w=12$. The results are shown in Table I and graphically in Fig. 4.

Worthwhile comparisons are difficult because earlier calculations contained such drastic simplyfying assumptions. For example, Karp *et al.* (1962) obtained for z, the depth of the oil-water interface, the formula

$$z=\{l^2+(l^2-d^2)\ln(r/r_e)/\ln(r_e/r_w)\}^{1/2}$$

This does not even give a change of maximum cone height with well depth of the same sign as our calculations, probably because of their assumption that the oil flow is due to radial pressure gradients only, vertical gradients being neglected.

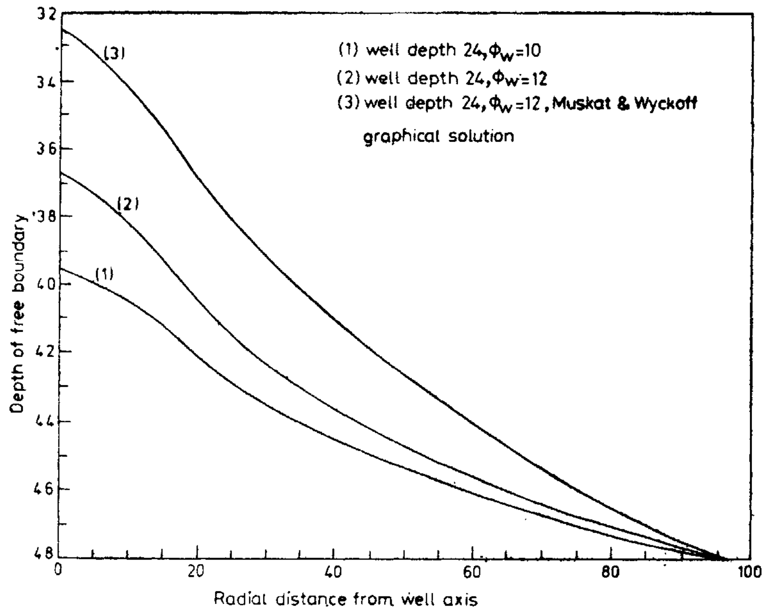

FIG. 4

Table 1

r	z_1	z_2	z_3
0.0	39.45756	36.78886	32.62
1.0	39.47705	36 82898	32.73
3.0	39.56367	37.00546	32.85
7.0	40.02601	37.84793	33.75
11.0	40.64606	38.81933	34.86
15.0	41.29300	39.73690	35.85
21.0	42.20564	40.94867	37.32
27.0	43.01736	41.98071	38.63
33.0	43.73171	42.86812	39.93
40.0	44.46056	43.76052	41.25
48.0	45.18090	44.63360	42.60
56.0	45.80512	45.38514	43.78
64.0	46.35249	46.04122	44.80
72.0	46.83727	46.62040	45.58
80.0	47.26995	47.13584	46.45
88.0	47.65750	47.59600	47.30
96.0	48.00000	48.00000	48.00

The calculated ordinates of the free boundary (z_1 and z_2) for the two different cases:

(1) $d \equiv 24$; $\phi_w = 10$

(2) $d = 24$; $\phi_w = 12$

with the parameters $\phi_e = 0$, $l = 48$, $r_w = 1$, and $r_e = 96$. Column 4 gives the calculated ordinates for case (2) using the graphical method.

However, at least qualitative agreement is obtained with some calculations by Muskat and Wyckoff (1935). They computed maximum cone height for $d = 12$ and $d = 24$ using a graphical method with the assumptions already referred to in section 1. Their technique has been used to calculate the oil-water interface shown in Fig. 4 for the case $d = 24$.

6. Discussion

Various ways of moving a free boundary have been tried by, for example, Garabedian (1956), Cryer (1970), Aitchison (1972), and Fox and Sankar (1973). The last was 'Regula-Falsi' a close relation of Newton's method, but treated the problem as an N-variable problem in the z_i, $i = 1, 2, \ldots, N$. It was thought that individual adjustment of boundary points might lead to an irregular shaped boundary and that some smoothing might be needed, but this has not proved to be the case. However, Fox and Sankar's problem of flow past a circular disc is currently being investigated using the present method, and early results seem to indicate that the weak singularity present does cause local oscillation.

The chief limitation of the method in its present form is that one coordinate of the free boundary must be a monotonic function of the other in order for the mesh to be defined.

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