Indian J. pure appl. Math., 13(5): 528–536, May 1982

ON TOTALLY REAL SUBMANIFOLDS OF A KÄHLERIAN MANIFOLD ADMITTING SEMI-SYMMETRIC METRIC F- CONNECTION

C. S. BAGEWADI*

Department of Mathematics, Karnatak University, Dharwad 580003

(*Received* 7 *October* 1980; *after revision* 30 *November* 1981)

A totally real submanifold of a Kähierian manifold admitting semi-symmetric metric F-connection is considered. It is shown that it admits semi-symmetric metric connection. Further the relations between the Christoffel symbols and semi-symmetric metric connection of the totally real submanifold induced by the semi-symmetric metric F-connection of the Kähierian manifold have been obtained. The conformally flatness or quasi-conformally flatness of totally real submanifold of Kähierian manifold is also considered. An equation involving Weyl conformal curvature tensor for a totally real submanifold to be recurrent whenever it is recurrent with respect to semi-symmetric metric connection is obtained.

INTRODUCTION

Let M^{2m}, $m\geqslant 2$ be a Kählerian manifold which admits special semi-symmetric metric F-connection. Yano and Imai (1975) have shown that M^{2m} is H-conformally flat, that is, Bochner curvature tensor is zero when the curvature tensor with respect to special semi-symmetric metric F-connection is zero. In this paper we consider a totally real submanifold M^n, $n>3$ of a Kählerian manifold M^{2m} admitting special semi-symmetric metric F-connection and show that M^n admits semi-symmetric metric connection and further we obtain the relations between the Christoffel symbols and semi-symmetric metric connection of M^n induced by the semi-symmetric metric F-connection of the Kählerian manifold M^{2m}.

Next we consider the conformally flatness or quasi-conformally flatness of M^n of M^{2m}. Amur and Desai (1975) have studied recurrent properties of projectively related spaces and they have obtained an equation involving the projective curvature tensor for the space (M, g) to be recurrent whenever projectively related space (M,g^*) is also recurrent. But here we obtain an equation involving Weyl conformal curvature tensor for a totally real submanifold M^n to be recurrent whenever it is recurrent with respect to semi-symmetric metric connection of M^n induced by the semi-symmetric metric F-connection of M^{2m}.

1. PRELIMINARIES

Let M^{2m}, $m\geqslant 2$ be a Kählerian manifold of real dimension $2m$ covered by a

* Department of Mathematics, P. G. Centre, B. R. Project, Shimoga (Karnatak).

system of co-ordinate neighbourhoods $\{U:x^h\}$ where here and in the sequel the indices h,i,j run over the range $\{1,2, \ldots 2m\}$ and let g_{ji}, F_i^j, ∇_j, K_{kji}^h, K_{ji} and R be the metric tensor, the complex structure tensor, the operator of covariant differentiation with respect to g_{ji}, the curvature tensor, the Ricci tensor and the scalar curvature of M^{2m} respectively.

Let M^n, $n>3$ be a Riemannian manifold of dimension n covered by a system of co-ordinate neighbourhoods $\{V:y^a\}$ where here and in the sequel the indices $a,b,c,\ldots$ run over the range $\{1,2,\ldots n\}$ and let g_{ab}, ∇_b, $K^d{}_{abc}$, K_{ab} and K be the metric tensor, the operator of covariant differentiation with respect to g_{ab}, the curvature tensor, the Ricci tensor and the scalar curvature of M^n respectively.

Assume that M^n is isometrically immersed in M^{2m} and represent the immersion by $x^h = x^h(y^a)$ and put $B_a^h = \partial_a^{R^h} \left(\partial_a = \frac{\partial}{\partial y^a} \right)$. Then we have

$$g_{ji}\, B_c^j\, B_b^i = g_{cb}. \qquad \ldots(1.1)$$

Definition 1.1 (Yano 1976)—If the transform by F_i^j of any tangent vector to M^n is normal to M^n then the submanifold M^n is said to be totally real in M^{2m}.

For a totally real submanifold we have

$$F_{ji} B_c^j\, B_b^i = 0. \qquad \ldots(1.2)$$

Let C_y^k $(x,y,z,\ldots = (n+1)(n+2),\ldots\ldots(2m))$ represent $(2m-n)$ mutually orthogonal unit vectors normal to M^n. The equations of Gauss and Weingarten with respect to Christoffel symbols are given by

$$\nabla\ B_c^h = H_{bc}^x\ C_x^h \qquad \ldots(1.3)$$

$$\nabla_b\ C_y^h = -\ H_{by}^c\ B_c^h \qquad \ldots(1.4)$$

where H_{bc}^x and $H_{by}^c\ =\ H_{be}^z\, g^{ec}\, g_{zy}$ are the second fundamental tensors of M^n with respect to normals C_y^k, g^{ec} being contravariant components of the metric tensor of M^n and g_{zy} the metric tensor of the normal bundle.

The normals C_x^h, $C_y^i \ldots.$ satisfy

$$B_c^j\, C_x^i\, g_{ji} = 0 \qquad \ldots(1.5)$$

$$C_y^j\, C_x^i\, g_{ji} = \delta_{yx} \text{ where } \delta_{yx} = \begin{cases} 1 \text{ if } x=y. \\ 0 \text{ if } x \neq y \end{cases} \qquad \ldots(1.6)$$

Definition 1.2—A Riemannian manifold M^n of a Kählerian manifold M^{2m} is said to be totally umbilical if the second fundamental tensor satisfies

$$H_{bc}^x = H^x g_{bc} \qquad \ldots(1.7)$$

where H^x being the scalar function.

The equations of Gauss for M^n with respect to Christoffel symbols are given by

$$K_{dcba} = K_{kjih} \; B^k_d \; B^j_c \; B^i_b \; B^h_a + A_{dcba} \qquad \text{...(1.8)}$$

where A_{dcba} is given by

$$A_{dcba} = H_{dax} \; H^x_{cb} - H_{cax} \; H^x_{db} \qquad \text{...(1.9)}$$

K_{dcba}, K_{kjih} are being the components of curvature tensors of M^n and M^{2m} respectively.

Let the Kählerian manifold M^{2m} admit semi-symmetric metric F-connection and be given by (Yano and Imai 1975)

$$\Gamma^h_{ji} = \left\{ {h \atop ji} \right\} + \delta^h_j \, p_i - g_{ji} \, p^h + F^h_j q_i + F^h_i q_j - F_{ji} q^h \qquad \text{...(1.10)}$$

where $\left\{ {h \atop ji} \right\}$ are the Christoffel symbols associated with the Riemannian connection of M^{2m} and p_i and q_i satisfy

$$q_i = F_{ti} \, p^t, \; p_i = -F_{ti} \, q^t p^h = p_t \, g^{th} \text{ and } q_i = q^t \, g_{ti} \qquad \text{...(1.11)}$$

The curvature tensors $\overset{\circ}{K}_{kjih}$ of Γ^h_{ji} and K_{kjih} of $\{ {h \atop ji} \}$ are related by

$$\overset{\circ}{K}_{kjih} = K_{kjih} - g_{kh} \, p_{ji} + g_{jh} \, p_{ki} - p_{kh} g_{ji} + p_{jh} g_{ki} - F_{kh} \, q_{ji} + F_{jh} \, q_{ki} - q_{kh} \; F_{ji}$$
$$+ \; q_{jh} \, F_{ki} + (\nabla_k q_j - \nabla_j \, q_k) \; F_{ih} - 2F_{kj} \; (p_i \; q_h - p_h q_i) \qquad \text{...(1.12)}$$

where

$$p_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \tfrac{1}{2} p_t p^t \; g_{ji} \qquad \text{...(1.13)}$$

and

$$q_{ji} = \nabla_j q_i - p_j q_i - p_i q_j + \tfrac{1}{2} p_t p^t \; F_{ji} \qquad \text{...(1.14)}$$

∇_j being the operator of covariant differentiation and consequently we have

$$q_{ji} = p_{jt} \; F^t_i, \; p_{ji} = -q_{jt} \; F^t_i, \; p^h_k = p_{kt} \; g^{th} \; \text{ and } \; q^h_k = q_{kt} \; g^{th}. \qquad \text{...(1.15)}$$

A semi-symmetric metric connection on M^n is defined by (Yano 1970)

$$\Gamma^a_{bc} = \left\{ {a \atop bc} \right\} + \delta^a_b \; p_c - g_{bc} \; p^a \qquad \text{...(1.16)}$$

where $\left\{ {a \atop bc} \right\}$ denotes the Christoffel symbols associated with the Riemannian connection of M^n, p^a is a covector field and $p^a = g^{ad} p_d$.

2. Lemmas

We shall prove the following Lemmas:

Lemma 2.1—Let M^n be a totally real submanifold semi-symmetric metric F-connection. Then M^n admits semi-symmetric metric connection.

Proof : As M^n is totally real submanifold of a Kählerian manifold M^{2m} with induced metric g_{ab} and induced Christoffel functions $\{ {a \atop bc} \}$, the relation between the connections by M^n and M^{2m} is given by

$$\Gamma^a_{bc} = (\partial_b \;\; B^h_c + \Gamma^h_{ji} \; B^j_c \; B^i_b) B^a_h \qquad \text{...(2.1)}$$

where Γ^a_{bc} is some connection on M^n and $B^h_c = \dfrac{\partial x^h}{\partial y^c}$ and Γ^h_{ji} is special semi-symmetric

metric F-connection on M^{2m} and

$$B_h^a = g^{ae} g_{uh} B_e^u . \quad \text{...(2.2)}$$

Using (1.10) in (2.1) we have

$$\Gamma_{bc}^a = \left[\partial_b B_c^h + \left(\left\{ {h \atop ji} \right\} + \delta_j^h p_i - g_{ji} p^h + F_j^h q_i + F_i^h q_j - F_{ji} q^h \right) B_c^j B_b^i \right] B_h^a . \quad \text{...(2.3)}$$

Using (1.1), (1.2), (2.2) and (2.1) in (2.3) we get

$$\Gamma_{bc}^a = \left\{ {a \atop bc} \right\} + \delta_b^a p_c - g_{bc} p^a \quad \text{...(2.4)}$$

where $p_c = B_c^h p^h$ and $p^a = g^{ab} p_b$.

Equation (2.4) shows that Γ_{bc}^a is semi-symmetric metric connection on M^n.

Lemma 2.2—Let M^n be totally real submanifold of a Kählerian manifold M^{2m} which admits semi-symmetric metric F-connection.
Then we have

$$\overset{\circ}{H}{}_{bc}^x = H_{bc}^x - \alpha^x (g_{bc} + L_{bc}) \quad \text{...(2.5)}$$

where H_{bc}^x and $\overset{\circ}{H}{}_{bc}^x$ are the second fundamental tensors with respect to the connection $\{ {a \atop bc} \}$ and the semi-symmetric metric connection Γ_{bc}^a of M^n induced by the semi-symmetric metric F-connection of M^{m} respectively and the tensor field L_{bc} of type (0,2) is defined by

$$\alpha_x L_{bc} = C_x^k F_{kj} B_b^j B_c^i F_{ti} \alpha^z C_z^t + C_x^k F_{ki} B_b^j B_c^i F_{sj} \alpha^w C_w^s \quad \text{...(2.6)}$$

α_x being some scalar function.

PROOF: The equations of Gauss and Weingarten with respect to connection $\left\{ {a \atop bc} \right\}$ are given by (1.3) and (1.4) respectively.

Since the Kählerian manifold M^{2m} admits semi-symmetric metric F-connection, so by Lemma 2.1, the totally real submanifold M^n admits semi-symmetric metric connection. The equations of Gauss and Weingarten with respect to semi-symmetric metric connection are given by

$$\overset{\circ}{\nabla}_b B_c^h = H_{bc}^y C_y^h \quad \text{...(2.7)}$$

$$\overset{\circ}{\nabla}_b C_y^h = - H_{by}^c B_c^h \quad \text{...(2.8)}$$

where $\overset{\circ}{\nabla}_b$ denotes the operator of covariant differentiation with respect to Γ_{bc}^a.

Decomposing the vector field p^h into its unique tangential and normal components along M^n we get

$$p^h = p^a B_a^h + \alpha^x C_x^h \text{ where } x = 1,2,\ldots\ldots, 2m—n. \quad \text{...(2.9)}$$

Multiply (2.7) by $C_h^k g_{kh}$ and use (1.6) to get

$$H_{bcx} = g_{kh}\ C^k_x\ \overset{\circ}{\nabla}_b B^h_c\,. \qquad \ldots(2.10)$$

Again we know that

$$\overset{\circ}{\nabla}_b\ B^h_c = \Gamma^h_{ji}\ B^j_b\ B^i_c - \Gamma^a_{bc}\ B^h_a + \partial_b B^h_c\,. \qquad \ldots(2.11)$$

Use of (1.10) and (1.16) after using (2.11) in (2.10) gives us as

$$\overset{\circ}{H}_{bcx} = g_{kh} C^k_x [\partial_b\ B^h_c + (\{^h_{ji}\} + \delta^h_j p_i - g_{ji}\, p^h + F^h_j q_i + F^h_i q_j - F_{ji} q^h)\ B^j_b\ B^i_c$$

$$- (\{^a_{bc}\} + \delta^a_b p_c - g_{bc} p^a) B^h_a]. \qquad \ldots(2.12)$$

Using the formulae (2.10) and (2.11) and equations (1.1), (1.2) and (1.5) in (2.12) we get

$$\overset{\circ}{H}_{bcx} = H_{bcx} - g_{kh}\ C^k_x g_{bc} p^h + g_{kh} C^k_x\ F^h_j\ q_i\ B^j_b\ B^i_c + g_{kh} C^k_x\ F^h_i q_j\ \ B^j_b\ B^i_c\,. \qquad \ldots(2.13)$$

Using (1.11) in (2.13) we get

$$\overset{\circ}{H}_{bcx} = H_{bcx} - g_{kh}\ C^k_x g_{bc}\ p^h + g_{kh}\ C^k_x\ F^h_j\ B^j_b\ B^i_c \times F_{ti} p^t + g_{kh} C^k_x\ F^h_i\ B^j_b\ B^i_c F_{sj}\ p^s \ldots(2.14)$$

Using (2.9), (1.1), (1.5) and (1.6) in (2.14) we get the required relation (2.5). Clearly from (2.6) we see that L_{bc} is a symmetric tensor field of type (0,2). Q.E.D.

Definition—A Riemannian manifold M^n is said to be L-Einstein if

$$L_{bc} = \frac{L}{n} g_{bc} \qquad \ldots(2.15)$$

where L_{bc} is defined by (2.6) and $L_{bc}\ g^{bc} = L$.

Since the totally real submanifold M^n of a Kählerian manifold admitting semi-symmetric metric F-connection admits semi-symmetric metric connection. The equation of Gauss for M^n with respect to semi-symmetric metric connection is given by

$$\overset{\circ}{K}_{dcba} = \overset{\circ}{K}_{kjih} B^k_d\ B^j_c\ B^i_b\ B^h_a + \overset{\circ}{A}_{dcba} \qquad \ldots(2.16)$$

where

$$\overset{\circ}{A}_{dcba} = \overset{\circ}{H}_{dax}\ \overset{\circ}{H}{}^x_{cb} - \overset{\circ}{H}_{cax}\ \overset{\circ}{H}{}^x_{db} \qquad \ldots(2.17)$$

$\overset{\circ}{K}_{dcba}$ and $\overset{\circ}{K}_{kjih}$ are being the components of curvature tensors of M^n and M^{2m} with respect to semi-symmetric metric connection and semi-symmetric metric F-connection respectively.

Let us set $\overset{\circ}{A}_{cb} = g_{da}\ \overset{\circ}{A}_{dcba},\ A_{cb} = g^{da}\ A_{dcba}$. $\qquad \ldots(2.18)$

and

$$\overset{\circ}{A} = g^{cb}\ \overset{\circ}{A}_{cb},\ A = g^{cb}\ A_{cb}. \qquad \ldots(2.19)$$

Lemma 2.3—Let M^n be a totally real submanifold of a Kählerian manifold M^{2m} which admits semi-symmetric metric F-connection.

Then

$$\overset{\circ}{A}_{dcba} - \frac{1}{(n-2)}(g_{da}\overset{\circ}{A}_{cb} - g_{db}\overset{\circ}{A}_{ca} + g_{cb}\overset{\circ}{A}_{da} - g_{ca}\overset{\circ}{A}_{db}) + \frac{\overset{\circ}{A}}{(n-1)(n-2)}(g_{da}g_{cb} - g_{ca}g_{db})$$
$$= A_{dcba} - \frac{1}{(n-2)}(g_{da}A_{cb} - g_{db}A_{ca} + g_{cb}A_{da} - g_{ca}A_{db})$$
$$+ \frac{A}{(n-1)(n-2)}(g_{da}g_{cb} - g_{ca}g_{db}). \qquad \ldots(2.20)$$

if M^n is totally umbilical and L-Einstein.

PROOF: Using (2.5) in (2.17) and the use of (1.9) will give us as follows

$$\overset{\circ}{A}_{dcba} = A_{dcba} - \alpha^x[H_{dax}g_{cb} - H_{cax}g_{db} + H_{cbx}g_{da} - H_{dbx}g_{ca}] - \alpha^x[H_{dax}L_{cb} - H_{cax}L_{db}$$
$$+ H_{cbx}L_{da} - H_{dbx}L_{ca}] + (\alpha_x\alpha^x)[g_{da}g_{cb} - g_{ca}g_{db}] + (\alpha_x\alpha^x)[L_{da}L_{cb} - L_{ca}L_{db}]$$
$$+ (\alpha_x\alpha^x)[g_{da}L_{cb} - g_{ca}L_{db} + g_{cb}L_{da} - g_{db}L_{ca}]. \qquad \ldots(2.21)$$

Transvecting (2.21) by g^{da} and using (2.18) we get

$$\overset{\circ}{A}_{cb} = A_{cb} - (n-2)\alpha^x H_{cbx} - \alpha^x H^b_{bx}g_{cb} + (n-2)(\alpha_x\alpha^x)L_{cb} + (\alpha_x\alpha^x)L\, g_{cb}$$
$$+ (n-1)(\alpha_x\alpha^x)g_{cb} + (\alpha_x\alpha^x)[L_{cb}L - L^t_c L_{tb}] - \alpha^x[H_{cbx}L - L^t_a H_{tbx}$$
$$+ H^b_{bx}L_{cb} - H_{ctx}L^t_b]. \qquad \ldots(2.22)$$

Again multiplying (2.22) by g^{cb} we get on using (2.19) as

$$\overset{\circ}{A} = A - 2(n-1)\alpha^x H^b_{bx} + n(n-1)(\alpha_x\alpha^x) + 2(n-1)(\alpha_x\alpha^x)\, L + (\alpha_x\alpha^x)(L^2 - |\, L\,|^2)$$
$$- 2\alpha^x\, (H^b_{bx}L - H_{tsx}\, L^{ts}) \qquad \ldots(2.23)$$

where $|\, L\,|^2 = L_{ab}\, L^{ab}$.

Using (2.21), (2.22)and (2.23) in the L.H.S. of (2.20) we get R.H S. if we use the definitions of totally umbilicity and L-Einstein i.e. eqns. (1.7) and (2.15) respectively.

3. THEOREMS

Let us set

$$X_{cb} = B^j_c\, B^i_b\, p_{ji} \text{ and } X = X_{cb}\, g^{cb}. \qquad \ldots(3.1)$$

where p_{ji} is defined by (1.13).

Definition 3.1—A manifold M^n is said to be X-Einstein if

$$X_{cb} = \frac{X}{n}\, g_{cb}. \qquad \ldots(3.2)$$

Definition 3.2—A Riemannian manifold M^n is said to be recurrent if

$$\nabla_1\, K^a_{dcb} = k_1\, K^a_{dcb}. \qquad \ldots(3.3)$$

where k_1 is some non-zero vector field.

The Weyl conformal curvature tensor with respect to semi-symmetric metric connection of a Riemannian manifold M^n, $n>3$ is given by

$$\overset{\circ}{C}_{dcba} = \overset{\circ}{K}_{dcba} - \frac{1}{(n-2)}[g_{da}\overset{\circ}{K}_{cb} - g_{ca}\overset{\circ}{K}_{db} + g_{cb}\overset{\circ}{K}_{da} - g_{db}\overset{\circ}{K}_{ca}] + \frac{\overset{\circ}{K}}{(n-1)(n-2)}(g_{da}g_{cb} - g_{ca}g_{db})$$
$$\ldots(3.4)$$

The W-tensor of M^n with respect to semi-symmetric metric connection is given by

$$\overset{o}{W}_{dcba} = a\overset{o}{K}_{dcba} + g_{da}\overset{o}{V}_{cb} - g_{ca}\overset{o}{V}_{db} + g_{cb}\overset{o}{V}_{da} - g_{ca}\overset{o}{V}_{db} \qquad ...(3.5)$$

where $\overset{o}{V}_{cb}$ is given by

$$\overset{o}{V}_{cb} = b\overset{o}{K}_{cb} - \overset{o}{K}\left(b + \frac{a}{2(n-1)} \right)\frac{g_{cb}}{n}. \qquad ...(3.6)$$

M^n is said to be quasi-conformally flat (Amur and Maralabhawi 1977) if $W=0$ and quasi-conformally flat with respect to semi-symmetric metric connection if $\overset{o}{W}=0$.

We shall prove the following theorems.

Theorem 3.1—Let M^n, $n>3$ be totally real submanifold of a Kählerian manifold M^{2m} which admits semi-symmetric metric F-connection. Then M^n is conformally flat if and only if M^n is conformally flat with respect to semi-symmetric metric connection of M^n induced by the semi-symmetric metric F-connection of M^{2m} provided M^n is totally umbilical and L-Einstein.

PROOF: Multiply (1.12) by $B^k_d\, B^j_c\, B^i_b\, B^h_a$ and using (1.1), (1.2), (1.8) and (3.1) we get

$$\overset{o}{K}_{dcba} = K_{dcba} + \overset{o}{A}_{dcba} - A_{dcba} - g_{da}X_{cb} + g_{ca}X_{db} - g_{cb}X_{da} + g_{db}X_{ca}. \qquad ...(3.7)$$

Multiplying (3.7) by g^{da} we have on using (3.1) as

$$\overset{o}{K}_{cb} = K_{cb} + \overset{o}{A}_{cb} - A_{cb} - Xg_{cb} - (n-2)X_{cb}. \qquad ...(3.8)$$

Again multiplying (3.8) by g^{cb} we have

$$\overset{o}{K} = K + \overset{o}{A} - A - 2(n-1)\,X. \qquad ...(3.9)$$

Using (3.7), (3.8) and (3.9) in (3.4) and the ultimate use of totally umbilicity and L-Einsteinness of M^n, that is, of Lemma 2.3 will give us the required result.

Theorem 3.2—Let M^n, $n>3$ be totally real submanifold of a Kählerian manifold M^{2m} which admits semi-symmetric metric F-connection. Then M^n is quasi-conformally flat if and only if M^n is quasi-conformally flat with respect to semi-symmetric metric connection of M^n induced by the semi-symmetric metric F-connection of M^{2m} provided M^n is totally umbilical, L-Einstein and X-Einstein.

PROOF: Using (3.7), (3.8) and (3.9) in (3.5) and again using (2.5), (1.8) and (1.9) and further the ultimate use of totally umbilicity, L-Einsteinness and X-Einsteinness will give us the required result.

Theorem 2.3—Let M^n, $n>3$ be totally real submanifold of a Kählerian manifold M^{2m} which admits semi-symmetric metric F-connection. A sufficient condition for M^n to be recurrent whenever it is recurrent with respect to semi-symmetric metric connection on M^n induced by the semi-symmetric metric F-connection of M^{2m} is that M^n is totally umbilical, L-Einstein and the Weyl conformal curvature tensor of M^n satisfies.

$$p_d C_{ecba} + p_e C_{deba} + p_b C_{dcea} + p_a C_{dcbe} = 0 \qquad ...(3.10)$$

PROOF: Differentiating covariantly on both sides of (3.7) with respect to semi-symmetric metric connection of M^n we have

$$\overset{\circ}{\nabla}_e \overset{\circ}{K}_{dcba} = \overset{\circ}{\nabla}_e K_{dcba} + \overset{\circ}{\nabla}_e \overset{\circ}{A}_{dcba} - \overset{\circ}{\nabla}_e A_{dcba} - g_{da}\overset{\circ}{\nabla}_e X_{cb} + g_{ca}\overset{\circ}{\nabla}_e X_{db} - g_{cb}\overset{\circ}{\nabla}_e X_{db} + g_{db}\overset{\circ}{\nabla}_e X_{ca} \quad ...(3.11)$$

where $\overset{\circ}{\nabla}$ denotes the operator of covariant differentiation with respect to semi-symmetric metric connection.

Using (1.16), $\overset{\circ}{\nabla}_e K_{dcba}$ can be written as

$$\overset{\circ}{\nabla}_e K_{dcba} = \nabla_e K_{dcba} - [p_d K_{ecba} + p_c K_{deba} + p_b K_{dcea} + p_a K_{dcbe}] + p^t[g_{ed} K_{tcba} + g_{ec} K_{dtba} + g_{eb} K_{dcta} + g_{ea} K_{dcbt}]. \quad ...(3.12)$$

Let M^n be recurrent with respect to semi-symmetric metric connection Γ^a_{bc}, then using (3.3) we have

$$\overset{\circ}{\nabla}_e \overset{\circ}{K}_{dcba} = k_e \overset{\circ}{K}_{dcba} \quad ...(3.13)$$

where $\overset{\circ}{k}_e$ is some non-zero vector field.

Using (3.11), (3.12), (3.13) and (3.7) we have

$$\begin{aligned}\overset{\circ}{k}_e K_{dcba} = {} & \nabla_e K_{dcba} - [p_d K_{ecba} + p_c K_{deba} + p_d K_{dcea} + p_a K_{dcbe}] + p^t[g_{ed} K_{tcba} + g_{ec} K_{dtba} \\ & + g_{eb} K_{dcta} + g_{ea} K_{dcbt}] + (\overset{\circ}{\nabla}_e \overset{\circ}{A}_{dcba} - \overset{\circ}{k}_e \overset{\circ}{A}_{dcba}) - (\overset{\circ}{\nabla}_e A_{dcba} - \overset{\circ}{k}_e A_{dcba}) \\ & - g_{da}(\overset{\circ}{\nabla}_e X_{cb} - \overset{\circ}{k}_e X_{cb}) + g_{ca}(\overset{\circ}{\nabla}_e X_{db} - \overset{\circ}{k}_e X_{db}) - g_{cb}(\overset{\circ}{\nabla}_e X_{da} - \overset{\circ}{k}_e X_{da}) \\ & + g_{db}(\overset{\circ}{\nabla}_e X_{ca} - \overset{\circ}{k}_e X_{ea}). \end{aligned} \quad ...(3.14)$$

Thus if

$$\begin{aligned} & -[p_d K_{ecba} + p_c K_{deba} + p_b K_{dcea} + p_a K_{dcbe}] + p^t[g_{ed} K_{tcba} + g_{ec} K_{dtba} + g_{eb} K_{dcta} + g_{ea} K_{dcbt}] \\ & + (\overset{\circ}{\nabla}_e \overset{\circ}{A}_{dcba} - \overset{\circ}{k}_e \overset{\circ}{A}_{dcba}) - (\overset{\circ}{\nabla}_e A_{dcba} - \overset{\circ}{k}_e A_{dcba}) - g_{da}(\overset{\circ}{\nabla}_e X_{cb} - \overset{\circ}{k}_e X_{cb}) \\ & + g_{ca}(\overset{\circ}{\nabla}_e X_{db} - \overset{\circ}{k}_e X_{db}) - g_{cb}(\overset{\circ}{\nabla}_e X_{da} - \overset{\circ}{k}_e X_{da}) + g_{db}(\overset{\circ}{\nabla}_e X_{ca} - \overset{\circ}{k}_e X_{ca}) = 0 \end{aligned} \quad ...(3.15)$$

then M^n is also recurrent with the same recurrence vector field $\overset{\circ}{k}_e$.

Multiplying (3.15) by g^{da} we get

$$\begin{aligned}(n-2)(\overset{\circ}{\nabla}_e X_{cb} - \overset{\circ}{k}_e X_{cb}) = {} & p^t(g_{ec} K_{tb} + g_{eb} K_{ct}) - (p_c K_{eb} + p_b K_{ce}) + (\overset{\circ}{\nabla}_e \overset{\circ}{A}_{cb} - \overset{\circ}{k}_e \overset{\circ}{A}_{cb}) \\ & - (\overset{\circ}{\nabla}_e A_{cb} - \overset{\circ}{k}_e A_{cb}) - g_{cb}(\overset{\circ}{\nabla}_e X - \overset{\circ}{k}_e X). \end{aligned} \quad ...(3.16)$$

Again multiplying (3.16) by g^{cb} we get

$$2(n-1)(\overset{\circ}{\nabla}_e X - \overset{\circ}{k}_e X) = (\overset{\circ}{\nabla}_e \overset{\circ}{A} - \overset{\circ}{k}_e \overset{\circ}{A}) - (\overset{\circ}{\nabla}_e A - \overset{\circ}{k}_e A). \quad ...(3.17)$$

Using (3.16) and (3.17) in (3.15) and the ultimate use of (2.20) will give us

$$p^t[g_{ed} C_{tcba} + g_{ec} C_{dtba} + g_{eb} C_{dcta} + g_{ea} C_{dcbt}] - [p_d C_{ecba} + p_c C_{deba} + p_b C_{dcea} + p_a C_{dcbe}] = 0. \quad ...(3.18)$$

Multiply (3.18) by g^{da} and using

$$g^{da}C_{deba}=g^{ba}C_{dcba}=g^{ca}C_{deba}=0,$$
$$C_{dcb}{}^{a}=C_{b a dc}, \quad C_{dcb}{}^{a}=-C_{cdba},$$
$$C_{dcba}=-C_{dcab}, \quad C_{dcba}=C_{cdab}$$

and $C_{dcba}+C_{cbda}+C_{bdca}=0$

we have $(n-3)p^{t}\ C_{dcbt}=0.$ ...(3.19)

Since $n>3$ we have

$p^{t}\ C_{dcbt}=0.$ ...(3.20)

Finally using (3.20) in (3.18) we get (3.10).

ACKNOWLEDGEMENT

The author is grateful to Prof. K. S. Amur for his kind encouragement.

REFERENCES

Amur, K., and Desai, P. (1975). Recurrent Properties of Projectively related spaces. *Tensor* (*N. S.*), **29**, 40–42.

Amur, K., and Maralabhawi, Y.B. (1977). On quasi-conformally flat spaces. *Tensor* (*N. S*), **31**, 194–98.

Yano, K., (1970). On semi-symmetric metric connection. *Revue Roum. Math. Pures Appl.*, **15**, 1579–86.

———(1976). Note on totally real submanifolds. *Tensor* (*N. S.*), **30**, 89-91.

Yano, K., and Imai, T. (1975). On semi-symmetric metric *F*-connections. *Tensor* (*N.S.*), 29, 134–38.