*Indian J. pure appl. Math.* 13(5): 537–544, May 1982

# ON $K$-CONTACT RIEMANNIAN AND SASAKIAN MANIFOLDS

R. S. MISHRA*

*Department of Mathematics, University of Kuwait, Kuwait*

(*Received* 19 *June* 1981; *after revision* 4 *November* 1981)

In this paper, identities in Riemann-Christoffel curvature tensors of a $K$-contact Riemannian manifold and also a Sasakian manifold have been obtained. Some of the consequences and applications of these identities have been studied.

## 1. INTRODUCTION

An odd dimensional differentiable manifold $V_n$, $n=2m+1$ on which there exist a tensor field $F$ of the type (1,1), a vector field $U$, a 1-form $u$ and a metric tensor $g$, satisfying

$$\text{(a) } F^2+I_n=u\otimes U, \text{ (b) } g(\bar{X},\bar{Y})=g(X,Y)-u(X)u(Y), \text{ (c) rank } ((F))=n-1 \quad \ldots(1.1a)$$

where

$$\bar{X} \underset{=}{\text{def}} FX \quad \ldots(1.2)$$

for all vector fields $X$, $Y$, $Z$, $\ldots \in T$ (tangent space to $V_n$ at a point $p$), is called an almost Grayan or an almost contact metric manifold. The structure $\{F,U,u,g\}$ is called an almost Grayan or an almost contact metric structure (Sasaki 1960, Hatakeyama *et al*. 1963).

On an almost Grayan manifold

$$\text{(a) } \bar{U}=0, \quad \text{(b) } u \circ F=0, \text{ (c) } u(U)=1, \quad \ldots(1.3)$$

$$\text{(a) } 'F(X,Y) \underset{=}{\text{def}} g(\bar{X},Y)=-'F(Y,X) \Leftrightarrow \text{(b) } g(\bar{X},Y)+g(X,\bar{Y})=0. \quad \ldots(1.4)$$

$'F$ is called the fundamental 2-form of $V_n$.

If on an almost Grayan manifold $V_n$,

$$\text{(a) } D_X U=\bar{X} \Leftrightarrow \text{(b) } (D_X u)(Y)='F(X,Y). \quad \ldots(1.5)$$

then $V_n$ is called $K$-contact Riemannian manifold (Okumura 1962).

It can be easily proved that on a $K$-contact Riemannian manifold

$$(D_Y\, 'F)(Z,W)='K(Z,W,Y,U)=u(K(Z,W,Y)) \quad \ldots(1.6)$$

where $K$ is the Riemann-Christoffel curvature tensor and

$$'K(X,Y,Z,W) \underset{=}{\text{def}} g(K(X,Y,Z),W).$$

A $K$-contact Riemannian manifold on which

$$(D_Y 'F)(Z,W)='K(Z,W,Y,U)=u(Z)g(W,Y)-u(W)g(Y,Z) \quad \ldots(1.7)$$

it satisfied, is called a Sasakian manifold (Sasaki 1965, 1967, 1968).

## 2. CURVATURE IDENTITIES ON A $K$-CONTACT RIEMANNIAN MANIFOLD

We have from (1.6) and (1.5a)

$$(D_X D_Y\, 'F)(Z,W)=(D_X 'K)(Z,W,Y,U)+'K(Z,W,D_X Y,U)+'K(Z,W,Y,\bar{X}),$$

$$-(D_Y D_X 'F)(Z,W)=-(D_Y\, 'K)(Z,W,X,U)-'K(Z,W,D_Y X,U)-'K(Z,W,X,\bar{Y}),$$

$$-(D_{[X,Y]}'F)(Z,W)=-'K(Z,W,[X,Y],U).$$

*Present Address: Road No. 8A, Rajendra Nagar, Patna 800016.

Adding the last three equations, we get

$$'K(X,Y,\overline{Z},W)+'K(X,Y,Z,\overline{W})+'K(X,\overline{Y},Z,W)+'K(\overline{X},Y,Z,W)$$
$$=(D_X'K)(Y,U,Z,W)+(D_Y'K)(U,X,Z,W). \quad ...(2.1a)$$

Applying Bianchi's second identities in this equation, we get

$$'K(X,Y,\overline{Z},W)+'K(X,Y,Z,\overline{W})+'K(\overline{X},Y,Z,W)+'K(X,\overline{Y},Z,W)+(D_U'K)(X,Y,Z,W)=0. \quad ...(2.1b)$$

Barring $W$ or $Z$ in this equation, we obtain

$$-'K(X,Y,\overline{Z},\overline{W})+'K(X,Y,Z,W)='K(\overline{X},Y,Z,\overline{W})+'K(X,\overline{Y},Z,\overline{W})+u(W)'K(X,Y,Z,U)$$
$$+(D_U'K)(X,Y,Z,\overline{W}),=-u(Z)'K(X,Y,W,U)+(D_U'K)(X,Y,\overline{Z},W)$$
$$+'K(\overline{X},Y,\overline{Z},W)+'K(X,\overline{Y},,\overline{Z},W). \quad ...(2.2a)$$

Barring $X$ or $Y$ in (2.1b), we get

$$'K(\overline{X},Y,\overline{Z},W)+'K(\overline{X},Y,Z,\overline{W})+'K(\overline{X},\overline{Y},Z,W)-'K(X,Y,Z,W)$$
$$=-u(X)'K(Z,W,Y,U)-(D_U'K)(\overline{X},Y,Z,W), \quad ...(2.2b)$$
$$'K(X,\overline{Y},\overline{Z},W)+'K(X,\overline{Y},Z,\overline{W})+'K(\overline{X},\overline{Y},Z,W)-'K(\overline{X,Y,Z,W})$$
$$=-u(Y)'K(Z,W,X,U)-(D_U'K)(X,\overline{Y},Z,W). \quad ...(2.2c)$$

(2.2a.c) also yield

$$'K(\overline{X},\overline{Y},\overline{Z},\overline{W})-'K(\overline{X},\overline{Y},Z,W)-'K(\overline{X},Y,Z,\overline{W})-'K(X,\overline{Y},Z,\overline{W})=-u(Y)'K(\overline{X},U,Z,\overline{W})$$
$$+u(X)'K(\overline{Y},U,Z,\overline{W})-u(W)'K(\overline{X},\overline{Y},\overline{Z},U)-(D_U'K)(\overline{X},\overline{Y},Z,\overline{W})...(2.2d)$$
$$'K(\overline{X},\overline{Y},\overline{Z},\overline{W})-K(X,Y,Z,W)-'K(X,Y,\overline{Z},\overline{W})-'K('X,Y,Z,W)$$
$$=-u(Z),K(\overline{X},\overline{Y},U,\overline{W})-u(W)'K(X,\overline{Y},\overline{Z},U)-u(Y)'K(\overline{Z},\overline{W},X,U$$
$$-(D_U'K)(X,\overline{Y},\overline{Z},\overline{W}). \quad ...(2.2e)$$

(2.1) and (2.2) are the identities for Riemann-Christoffel curvature tensor on a $K$-contact Riemannian manifold.

It may be noted from (2.2a) that we have on a $K$-contact Riemannian manifold

$$u(Z)'K(X,Y,W,U)+u(W)'K(X,Y,Z,U)+'K(\overline{X},Y,Z,\overline{W})+'K(X,\overline{Y},Z\overline{W})$$
$$=(D_U'K)(X,Y,\overline{Z},W)-(D_U'K)(X,Y,Z,\overline{W})+'K(\overline{X}\overline{Y},Z,W)+'K(X,\overline{Y},Z\overline{W}) \quad ...(2.3)$$

Equation (2.1b) is equivalent to

$$K(X,Y,\overline{Z})-\overline{K(X,Y,Z)}+K(\overline{X},Y,Z)+'K(X,\overline{Y},Z)+(D_UK)(X,Y,Z)=0.$$

Contracting this equation, with regard to $X$, we obtain

$$\mathrm{Ric}(Y,\overline{Z})+\mathrm{Ric}(\overline{Y},Z)+(D_U\mathrm{Ric})(Y,Z)=0. \quad ...(2.4a)$$

Barring $Y$ or $Z$ in this equation, we obtain

$$\mathrm{Ric}(\overline{Y},\overline{Z})-\mathrm{Ric}(Y,Z)+u(Y)\mathrm{Ric}(Z,U)+(D_U\mathrm{Ric})(\overline{Y},Z)=0, \quad ...(2.4b)$$
$$\mathrm{Ric}(\overline{Y},\overline{Z})-\mathrm{Ric}(Y,Z)+u(Z)\mathrm{Ric}(Y,U)+(D_U\mathrm{Ric})(Y,\overline{Z})=0. \quad ...(2.4c)$$

Thus on a $K$-contact Riemannian manifold

$$u(Y)\mathrm{Ric}(Z,U)-u(Z)\mathrm{Ric}(Y,U)+(D_U\mathrm{Ric})(\overline{Y},Z)-(D_U\mathrm{Ric})(Y,\overline{Z})=0. \quad ...(2.5)$$

The equation (2.4a) is equivalent to

$$-\overline{RY}+R\overline{Y}+(D_UR)Y=0, \quad ...(2.6a)$$

where

$$g(RY,Z) \underset{=}{\mathrm{def}} \mathrm{Ric}(Y,Z). \quad ...(2.6b)$$

Contracting (2.6a), we obtain $Ur=0$, where $r$ is the scalar curvature of $V_n$. Hence we have the following theorem:

*Theorem* 2.1—The scalar curvature of a $K$-contact Riemannian manifold is constant along the contact vector $U$.

### 3. Properties on a $K$-contact Riemannian Manifold

*Theorem* 3.1—If a $K$-contact Riemannian manifold is conformally flat, then

$$g(Y,W)(D_U\text{Ric})(X,Z)-g(Y,Z)(D_U\text{Ric})(X,W)-g(X,W)(D_U\text{Ric})(Y,Z) + g(X,Z)(D_U\text{Ric})(Y,W)+(n-2)(D_U{}'K)(X,Y,Z,W)=0. \quad ...(3.1)$$

Proof : We have

$$'K(X,Y,Z,W) = \frac{1}{n-2}\{g(X,W)\text{ Ric }(Y,Z)-g(X,Z)\text{ Ric}(Y,W)-g(Y,W)\text{ Ric}(X,Z) + g(Y,Z)\text{ Ric}(X,W)\}-\frac{r}{(n-1)(n-2)}\{g(X,W)g(Y,Z)-g(Y,W)g(X,Z)\}.$$

Substituting from this equation in (2.1b). we obtain (3.1).

### 4. Curvature Identities on a Sasakian Manifold

From (1.7) and (1.5b), we have

$$(D_XD_Y{}'F)(Z,W)={}'F(X,Z)g(Y,W)-{}'F(X,W)g(Y,Z)+u(Z)g(W,D_XY)-u(W)g(Z,D_XY).$$

Consequently, we get as in § 2

$$'K(X,Y,Z,\bar{W})+{}'K(X,Y,\bar{Z},W)={}'F(X,Z)g(Y,W)-{}'F(Y,Z)g(X,W) -{}'F(X,W)g(Y,Z)+{}'F(Y,W)g(X,Z). \quad ...(4.1a)$$

Barring $W$ in this equation and using (1.7), we obtain

$$'K(X,Y,\bar{Z},\bar{W})-{}'K(X,Y,Z,W)={}'F(Z,X){}'F(Y,W)+{}'F(Y,Z){}'F(X,W) -g(Y,Z)g(X,W)+g(X,Z)g(Y,W). \quad ...(4.1b)$$

Since $'K$ is symmetric in two pairs of slots, we have from the above equation

$$'K(\bar{X},\bar{Y},Z,W)={}'K(X,Y,\bar{Z},\bar{W}). \quad ...(4.2a)$$

Consequently

$$'K(\bar{X},\bar{Y},\bar{Z},\bar{W})={}'K(X,Y,Z,W)+u(X)\{u(Z)g(Y,W)-u(W)g(Y,Z)\} -u(Y)\{u(Z)g(X,W)-u(W)g(X,Z)\}. \quad ...(4.2b)$$

From (4.1), we also get

$$'K(X,\bar{Y},Z,\bar{W})+{}'K(X,\bar{Y},\bar{Z},W)={}'F(X,Z){}'F(Y,W)-{}'F(Y,Z){}'F(X,W) +g(\bar{Y},\bar{Z})g(X,W)-g(\bar{Y},\bar{W})g(X,Z), \quad ...(4.3a)$$

$$'K(\bar{X},Y,Z,\bar{W})+{}'K(\bar{X},Y,\bar{Z},W)={}'F(Y,W){}'F(X,Z)-{}'F(Y,Z){}'F(X,W) + g(\bar{X},\bar{W})g(Y,Z)-g(\bar{X},\bar{Z})g(Y,W). \quad ...(4.3b)$$

(4.1 a,b), (4.2) and (4.3) are identities in Riemann-Christoffel curvature tensor on a Sasakian manifold.

Interchanging $X$ and $Z$ and $Y$ and $W$ in (4.1a), we obtain

$$-{}'K(X,\bar{Y},Z,W)-{}'K(\bar{X},Y,Z,W)={}'K(X,Y,Z,\bar{W})+{}'K(X,Y,\bar{Z},W). \quad ...(4.1c)$$

*Theorem* 4.1—We have on a Sasakian manifold.

$$'K(X,Y,\bar{Z},\bar{W})+{}'K(Y,Z,\bar{X},\bar{W})+{}'K(Z,X,\bar{Y},\bar{W})=2\{{}'F(X,Y){}'F(Z,W) +{}'F(Y,Z){}'F(X,W)+{}'F(Z,X){}'F(Y,W)\}. \quad ...(4.4)$$

PROOF : Using (4.1b) in Bianchi's cyclic identities, we obtain (4.4).

*Theorem* 4.2—Let us put

$$'H(Y,Z) \underline{\underline{\text{def}}}\ tr\ \overline{K(Y,Z)} \qquad \text{...(4.5a)}$$

$$*H(Y,Z) \underline{\underline{\text{def}}}\ -'H(Y,\overline{Z}). \qquad \text{...(4.5b)}$$

Then

$$*H(Y,Z)-\text{Ric}(Y,Z)=g(\overline{Y},\overline{Z})-(n-1)g(Y,Z)=-u(Y)u(Z)-(n-2)g(Y,Z). \qquad \text{...(4·5c)}$$

Consequently $*H$ is symmetric.

PROOF : Equation (4.1b) is equivalent to

$$-\overline{K(X,Y,\overline{Z})}-K(X,Y,Z)='F(Z,X)\overline{Y}-'F(Z,Y)\overline{X}-g(Y,Z)X+g(X,Z)Y.$$

Contracting this equation with respect to $X$ and using (4.5a,b), we get (4.5c)

*Theorem* 4.3—Let us put further

$$\text{(a)}\ g(HX,Y) \underline{\underline{\text{def}}} *H(X,Y), \quad \text{(b)}\ *h \underline{\underline{\text{def}}}\ tr\ H. \qquad \text{...(4.6)}$$

Then on a Sasakian manifold, $(n>1)$

$$r>*h. \qquad \text{...(4.7)}$$

PROOF : (4.5c) is equivalent to

$$HY-RY=-u(Y)U-(n-2)Y.$$

Contraction of this equation and use of (4.6) yields

$$*h-r=-(n-1)^2.$$

This equation proves the statement.

*Note* 4.1: It may be noted that (4.5b,c) and (4.1) yield the known results

$$\left.\begin{array}{ll}\text{(a)}\ *H(X,U)=0, & \text{(b)}\ \text{Ric}\,(X,U)=(n-1)u(X), \\ \text{(c)}\ 'K(\overline{X},\overline{Y},Z,U)=0. & \end{array}\right\} \qquad \text{...(4.8)}$$

*Theorem* (4.4)—We have on a Sasakian manifold;

$$\left.\begin{array}{l}\text{(a)}\ \text{Ric}\,(\overline{Y},\overline{Z}) = \text{Ric}\,(Y,Z)-(n-1)\,u\,(Y)\,u\,(Z), \\ \text{(b)}\ \text{Ric}\,(\overline{Y},Z)+\text{Ric}\,(Y,\overline{Z})=0.\end{array}\right\} \qquad \text{...(4.9)}$$

PROOF : Equation (4.2 b) yields

$$-K\,(X,Y,Z\,) = K(X,Y,Z) + u\,(X)\,u\,(Z)\,Y - g\,(Y,Z)\,u\,(X)\,U$$
$$- u\,(Y)\,u\,(Z)\,X + u\,(Y)\,g\,(X,Z)\,U.$$

Contraction of this equation with respect to $X$ yields (4.9 a). (4.9 b) follows from (4. 9 a).

*Note* 4.2 : (4.9) is well known, but has been obtained here by a different and simpler method.

*Theorem* 4.5 — We have on a Sasakian manifold

$$*H\,(\overline{Y},\overline{Z}) = *H\,(Y,Z). \qquad \text{...(4.10)}$$

PROOF : Barring $Y$, $Z$ in (4.5) $c$, we immediately get (4.10).

## 5. HOLOMORPHIC SECTIONAL CURVATURE

Holomorphic sectional curvature of an almost Grayan manifold $V_n$ at a point $p$ in the direction $X$ is given by

$$k = \frac{-\,'K\,(X,\overline{X},X,\overline{X}\,)}{g\,(X,X\,)\,g\,(\overline{X},\overline{X})}\ .$$

Shukla has proved that if an almost Grayan manifold is of constant holomorphic sectional curvature at $p$, then

$$
\begin{aligned}
&6\,'K(X,Y,Z,W) + 6\,'K(\bar{X},\bar{Y},Z,W) + 6\,'K(X,Y,\bar{Z},\bar{W}) + 6\,K(\bar{X},\bar{Y},\bar{Z},\bar{W}) \\
&4\,'K(\bar{Y},\bar{Z},W,X) + 4\,'K(Y,Z,\bar{W},\bar{X}) + 4\,'K(\bar{X},\bar{Z},Y,W) + 4\,'K(X,Z,\bar{Y},\bar{W}) \\
&-2\,'K(X,\bar{Y},\bar{Z},W) - 2\,'K(X,\bar{Y},Z,\bar{W}) - 2\,'K(\bar{X},Y,Z,\bar{W}) - 2\,'K(\bar{X},Y,\bar{Z},W) \\
&+ u(Z)\,\{3'K(X,Y,W,U) + 3'K(\bar{X},\bar{Y},W,U) + 'K(\bar{W},\bar{Y},X,U) + 'K(\bar{X},\bar{W},Y,U) \\
&+ 'K(\bar{W},X,\bar{Y},U) + 'K(Y,\bar{W},\bar{X},U) - 2u(Y)\,'K(X,U,W,U) + 2u(X)\,'K(Y,U,W,U\}) \\
&-u(W)\,\{3'K(X,Y,Z,U) + 3'K(\bar{X},\bar{Y},Z,U) + 'K(\bar{Z},\bar{Y},X,U) + 'K(\bar{X},\bar{Z},Y,U) \\
&+ 'K(\bar{Z},X,\bar{Y},U) + 'K(Y,\bar{Z},\bar{X},U) - 2u(Y)\,'K(X,U,Z,U) + 2u(X)\,'K(Y,U,Z,U)\} \\
&+u(X)\,\{3'K(Z,W,Y,U) + 3'K(\bar{Z},\bar{W},Y,U) + 'K(\bar{Y},\bar{W},Z,U) + 'K(\bar{Z},\bar{Y},W,U) \\
&+ 'K(\bar{Y},Z,\bar{W},U) + 'K(W,\bar{Y},\bar{Z},U)\} \\
&-u(Y)\,\{3'K(Z,W,X,U) + 3'K(\bar{Z},\bar{W},X,U) + 'K(\bar{X},\bar{W},Z,U) + 'K(\bar{Z},\bar{X},W,U) \\
&+ 'K(\bar{X},Z,\bar{W},U) + 'K(W,\bar{X},\bar{Z},U)\} \\
&= -2k\,\{4g(\bar{X},\bar{Z})\,g(\bar{Y},\bar{W}) - 4g(\bar{X},\bar{W})\,g(\bar{Y},\bar{Z}) + 8\,'F(X,Y)\,'F(Z,W) \\
&- 4\,'F(Y,Z).\,'F(X,W) + 4\,'F(Y,W)\,'F(X,Z) + u(X)\,u(Z)\,g(Y,W) \\
&- u(Y)\,u(Z)\,g(X,W) + u(Y)\,u(W)\,g(X,Z) - u(X)\,u(W)\,g(Y,Z)\}. \quad ..(5.1)
\end{aligned}
$$

When the manifold is *K*-contact Riemannian, it is not possible to simplify the above equation further. But when the manifold is Sasakian, the above equation can be simplified as seen from the following theorem :

*Theorem* 5.1—When a Sasakian manifold is of constant holomorphic sectional curvature $k$, then

$$k = 1 \quad ...(5.2)$$

and

$$'K(X,Y,Z,W) = g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W) - 'F(X,Y)\,'F(Z,W). \quad ...(5.3)$$

PROOF : Substituting from (4.1b), (4.2), (4.3) and (1.7) in (5.1), we obtain

$$
\begin{aligned}
16'K(X,Y,Z,W) = {}&4'F(X,Z)\,'F(Y,W) - 4'F(Y,Z)\,'F(X,W) - 8'F(X,Y)\,'F(Z,W) \\
&+ 12g(Y,Z)\,g(X,W) - 12g(X,Z)\,g(Y,W) - 3u(X)u(Z)\,g(Y,W) \\
&+ 3u(Y)\,u(Z)\,g(X,W) + 3u(X)\,u(W)\,g(Y,Z) \\
&- 3u(Y)\,u(W)\,g(X,Z) \\
&- k\,\{4g(X,Z)\,g(Y,W) - 4g(X,W)\,g(Y,Z) \\
&+ 8'F(X,Y)\,'F(Z,W) - 4'F(Y,Z)\,'F(X,W) + 4'F(Y,W)\,'F(X,Z) \\
&- 3u(X)\,u(Z)\,g(Y,W) + 3u(Y)\,u(Z)\,g(X,W) \\
&+ 3u(X)\,u(W)\,g(Y,Z) - 3u(Y)\,u(W)\,g(X,Z)\}. \quad ...(5.4)
\end{aligned}
$$

whence

$$
\begin{aligned}
16'K(X,Y,\bar{Z},\bar{W}) - 16'K(X,Y,Z,W) = {}&16'F(Y,Z)\,'F(X,W) - 16'F(X,Z)\,'F(Y,W) \\
&+ 16g(X,Z)\,g(Y,W) - 16g(Y,Z)\,g(X,W) \\
&- (k-1)\,\{u(X)\,u(Z)\,g(Y,W) - u(Y)\,u(Z)\,g(X,W) \\
&- u(X)\,u(W)\,g(Y,Z) + u(Y)\,u(W)\,g(X,Z)\}.
\end{aligned}
$$

Comparison of this equation with (4.1b) yields (5.2). Putting $k=1$ in (5.4), we obtain (5.3).

*Corollary* 5.1—On a Sasakian manifold of constant holomorphic sectional curvature

$$\text{Ric}\,(Y,Z) = n\,g\,(Y,Z) - u\,(Y)\,u\,(Z), \qquad \text{...(5.4a)}$$

$$r = n^2 - 1. \qquad \text{...(5.4b)}$$

Hence the scalar curvature $r$ is always positive.

PROOF : (5.3) is equivalent to $K(X,Y,Z) = g\,(Y,Z)X - g(X,Z)Y - {}'F(X,Y)\bar{Z}$. Contracting this equation, we obtain $\text{Ric}\,(Y,Z) = (n-1)\,g(Y,Z) + g(\bar{Y},\bar{Z})$ which is equivalent to (5.4a). From (5.4a) $RY = nY - u(Y)U$. Contracting this equation, we obtain (5.4b).

*Corollary* 5.2—On a Sasakian manifold of constant holomorphic sectional curvature $\text{Ric}\,(Y,Y) + \| u(Y) \|^2$ , is positive definite.

The statement follows from (5.4a).

*Corollary* 5.3—A Sasakian manifold of constant holomorphic sectional curvature cannot be an Einstein manifold.

The statement is obvious from (5.4a).

## 6. Properties on a Sasakian Manifold

$C$-Bochner curvature tensor $'B$ on a Sasakian manifold $V_n$ is given by (Matsumoto and Chuman 1969)

$$\begin{aligned} 'B(X,Y,Z,W) &= {}'K(X,Y,Z,W) + \frac{1}{n+3}\{\text{Ric}\,(X,Z)g\,(Y,W) - \text{Ric}\,(Y,Z)\,g\,(X,W) \\ &+ g\,(X,Z)\,\text{Ric}\,(Y,W) - g\,(Y,Z)\,\text{Ric}\,(X,W) + \text{Ric}\,(\bar{X},Z)\,{}'F(Y,W) \\ &- \text{Ric}\,(\bar{Y},Z){}'F(X,W) + {}'F(X,Z)\,\text{Ric}\,(\bar{Y},W) - {}'F(Y,Z)\,\text{Ric}\,(\bar{X},W) \\ &+ 2\,\text{Ric}\,(\bar{X},Y){}'F(Z,W) + 2{}'F(X,Y)\,\text{Ric}\,(\bar{Z},W) - \text{Ric}\,(X,Z)u(Y)u(W) \\ &+ \text{Ric}\,(Y,Z)u(X)u(W) - u(X)u(Z)\,\text{Ric}\,(Y,W) + u(Y)u(Z)\,\text{Ric}\,(X,W)\} \\ &- \frac{l+n-1}{n+3}\{{}'F(X,Z){}'F(Y,W) - {}'F(Y,Z){}'F(X,W) + 2{}'F(X,Y){}'F(Z,W)\} \\ &+ \frac{l}{n+3}\{g(X,Z)u(Y)u(W) + g(Y,W)u(X)u(Z) - g(Y,Z)\,u(X)u(W) \\ &- g(X,W)u(Y)u(Z)\} - \frac{l-4}{n+3}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}, \end{aligned} \qquad \text{...(6.1a)}$$

where $l \overset{\text{def}}{=} \dfrac{r+n-1}{n+1}$ . ...(6.1b)

It can be easily proved that

$$\left.\begin{aligned} &\text{(a)}\quad 'B(X,Y,\bar{Z},\bar{W}) = {}'B(\bar{X},\bar{Y},Z,W) \\ &\text{(b)}\quad 'B(\bar{X},\bar{Y},\bar{Z},\bar{W}) = {}'B(X,Y,Z,W) \\ &\text{(c)}\quad B(X,Y,Z,U) = 0 \end{aligned}\right\} \qquad \text{...(6.2)}$$

*Theorem* 6.1—A Sasakian manifold cannot be flat.

PROOF : If a Sasakian manifold is flat, it is Ricci flat. In that case (4.9a) assumes the form $(n-1)\,u\,(Y)u\,(Z) = 0$, which is not possible, since $n>1$.

*Theorem* 6.2—If a Sasakian manifold is an Einstein manifold, then its scalar curvature is $n\ (n-1)$.

PROOF : If a Sasakian manifold is an Einstein manifold, we have from (4.9a)

$$\frac{r}{n} g\,(\overline{Y},\overline{Z}) = \frac{r}{n} g\,(Y,Z) - (n-1)\ u(Y)u(Z).$$

This equation yields the result.

*Theorem* 6.3—If a Sasakian manifold is of constant Riemannian curvature $k$, then $k$ must be unity and $r = n\ (n-1)$.

PROOF : We have $K(X, Y,Z) = k\ \{g(Y,Z)X - g(X,Z)Y\}$.

Barring $X$ and $Y$ in the above equation, using (4.2a) and (4.1b) and subtracting the above equation from the resulting equation, we obtain

$$(k-1)\ \{'F(Y,Z)\overline{X} - 'F(X,Z)\overline{Y} - g(Y,Z)X + g(X,Z)Y\} = 0$$

$$(k-1)\ \{-u(Y)X + u(X)\,Y\} = 0 \Rightarrow (k-1)(n-1)u(Y) = 0 \Rightarrow k = 1.$$

Since a manifold of constant Riemannian curvature is an Einstein manifold, we have $r = n\ (n-1)$.

*Note* 6.1 : It is well known (Mishra 1973) that a necessary and sufficient condition for $V_n$, $n > 2$ to be of constant Riemannian curvature is that it is projectively flat. Therefore, in the above theorem the words 'of constant Riemannian curvature' may be replaced by 'projectively flat'.

*Note* 6.2 : Theorems (6.2) and (6.3) are known, but they have been obtained here by a different method.

*Theorem* 6.4—On a Sasakian manifold $'K$ is parallel along $U$.

PROOF : We have $'K(Z, W, Y, U) = u(Z)g\ (W,Y) - u(W)g(Y,Z)$.

Consequently

$$(D_X{'K})(Z,W,Y,U) = -{'K}(Z,W,Y,\overline{X}) + {'F}(X,Z)g(W,Y) - {'F}(X,W)g(Y,Z).$$

Hence

$$\begin{aligned}(D_U{'K})(X,Y,Z,W) &= -(D_X{'K})(Y,U,Z,W) - (D_Y{'K})(U,X,Z,W)\\ &= -{'K}(Z,W,\overline{X},Y) - {'K}(Z,W,X,\overline{Y}) - {'F}(X,Z)g(Y,W)\\ &\quad + {'F}(Y,Z)g(X,W) + {'F}(X,W)g(Y,Z) - {'F}(Y,W)g(X,Z)\\ &= 0\end{aligned}$$

by virtue of (4.1a). Hence, we have the statement.

*Theorem* 6.5—A necessary and sufficient condition for a Sasakian manifold to be an Einstein manifold is

$$^*H(Y,Z) = g(\overline{Y},\overline{Z}). \qquad \text{.. (6.3)}$$

PROOF : When the Sasakian manifold is an Einstein manifold, we have from Theo. (6.2)

$$\text{Ric} = (n-1)g. \qquad \text{...(6.4)}$$

Substituting from this equation in (4.5c), we have (6.3).

Conversely, if (6.3) is satisfied, (4.5c) yields (6.4).

*Theorem* 6.6—On a Sasakian manifold $^*H$ can never be equal to Ric.

PROOF : Putting

$$^*H = \text{Ric},$$

in (4.5c), we get $u(Y)u(Z) + (n-2)g(Y,Z) = 0$, which is impossible.

*Theorem* 6.7—Let us put

$$Q(X) \underline{\underline{\text{def}}} -'K(X,\bar{X},X,\bar{X}). \qquad \ldots(6.5)$$

Then

$$Q(X+\bar{Y}) + Q(X-\bar{Y}) + Q(X+Y) + Q(X-Y) - 4Q(X) - 4Q(Y)$$
$$= -16'K(X,\bar{X},Y,\bar{Y}) - 8\,|\,'F(X,Y)\,|^2 + 8\,\|X\|^2\,\|Y\|^2 - 8\,|\,g(X,Y)\,|^2$$
$$-4\,|u(X)|^2\,\|Y\|^2 - 4\,|u(Y)|^2\,\|X\|^2 + 8u(X)u(Y)g(X,Y), \qquad \ldots(6.6a)$$

$$3Q(X+\bar{Y}) + 3Q(X-\bar{Y}) - Q(X+Y) - Q(X-Y) - 4Q(X) - 4Q(Y)$$
$$= -32'K(X,Y,X,Y) + 24\,|\,'F(X,Y)\,|^2 + 24\,|\,g(X,Y)\,|^2$$
$$-24\,\|X\|^2\,\|Y\|^2 - 4\,|\,u(X)\,|^2\,\|Y\|^2 + 8u(X)u(Y)g(X,Y)$$
$$-6|u(Y)|^2\,\|X\|^2 + 2|\,u(X)\,|^2\,|u(Y)|^2 - 2|\,u(Y)\,|^2\,\|X\|^2 + 2|u(Y)|^4. \qquad \ldots(6.6b)$$

PROOF : (6.6a,b) follow by direct computation and use of (1.1a), (4.1b), (4.2) and (4.3).

*Definition* 6.1—If on a $K$-contact Riemannian manifold

$$\text{Ric} = ag + b\, u \otimes u, \quad a + b = 2m > 2, \qquad \ldots(6.7)$$

then the $K$-contact Riemannian manifold is called $C$-Einstein or $\eta$-Einstein manifold.

*Theorem* 6.8—On a $C$-Einstein manifold, Ric is parallel along $U$.

PROOF : Substituting from (6.7) in (2.4a), we get

$$D_U \text{Ric} = 0,$$

which proves the statement.

It may be noted that on a $C$-Einstein manifold, (4.9) are identically satisfied.

## REFERENCES


Hatakeyama, Y., Ogawa, Y., and Tanno, S. (1963). Some properties of manifolds with almost metric structure. *Tohoku Math. J.*. 15 (1), 42–48.

Matsumoto, M.. and Chuman, G. (1969). On $C$-Bochner curvature tensor. *T.R.U. Math. Japan*, **5**, 21–30.

Mishra, R. S. (1973). A course in tensors with applications to Riemannian Geometoy. Pothushala Pvt. Ltd., Allahabad.

——— (1982). Structures on differentiable manifolds and their applications. (To be published).

Okumura, M. (1962). Some remarks in spaces with certain contact structures. *Tohoku Math. J.*, **14**, 135–45.

Sasaki, S. (1960). On differentiable manifolds with certain structures which are closely related to almost contact structure. *Tohoku Math. J.*, **12**, 456-76.

———(1965, 1967, 1968). Almost contact manifolds, I, II, III. A lecture note, Tohoku University.

Shukla, A. Ph. D. Thesis (Private Communication).