Indian J. pure appl. Math., 13(5) : 545–550 May 1982

H-FUNCTIONS AND LARGE DEFLECTION OF A CIRCULAR PLATE

A. K. SINGH*, Y. N. PRASAD AND A. SIDDIQUI

Applied Mathematics Section, Institute of Technology, Banaras Hindu University
Varanasi 221005

(*Received* 24 *November* 1980; *after revision* 29 *August* 1981)

The present paper deals with large deflection analysis and determination of the deflections and bending stresses for a clamped circular plate under non-uniform force distribution. The technique, used here, is the Berger's (1955) approximate method. The load shape has been assumed in the form of an arbitrary involving the Fox's (1961) H-function. On account of a very general nature of this H-function, the solution obtained here yields many useful and interesting results. The case when the plate undergoes small deflection has been treated as special case of large deflection. In the last section of the paper the behaviour of a family of load shapes has been discussed.

1. INTRODUCTION

Since a vast literature is readily available on Fox's H-function, we shall omit details and enumerate here the results which are useful in the present discussion.

Fox's H-function is defined and represented as

$$H[x]=H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right]=\frac{1}{2\pi\omega}\int_L \phi(s)x^s\,ds, \quad \omega=\sqrt{-1}, \qquad \ldots(1.1)$$

where

$$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}. \qquad \ldots(1.2)$$

The conditions on various parameters and nature of the contour L have been given in detail in Fox (1961) and Braaksma (1964). If we let

$$V=\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j \qquad \ldots(1.3)$$

then for $V>0$, the integral (1.1) is absolutely convergent and defines the H-function analytic in the sector $|\arg x|<\frac{1}{2}\pi V$. Braaksma (1963) has shown that

$H[x]=O(|x|^{\bar{\alpha}})$ for max $|x|\to 0$,

* Supported by the Council of Scientific and Industrial Research, New Delhi under grant No. 7/13 (1195)/78 – EMR—I.

where

$$\bar{\alpha} = \min \operatorname{Re} (b_j/\beta_j), \quad j=1,\ldots,m \qquad \ldots(1.4)$$

and

$$H[x] = O(|x|^{\bar{\beta}}) \text{ for } \min |x| \to \infty,$$

where

$$\bar{\beta} = \max \operatorname{Re} [(a_j - 1)/\alpha_j], \; j=1,\ldots,n. \qquad \ldots(1.5)$$

Following two integrals will be used in the sequel:

$$\int_0^1 y^\rho (1-y)^\mu H[z(y-1)^\delta] dy$$

$$= \Gamma(\rho+1) \; H^{m,n+1}_{p+1,q+1}\left[(-1)^\delta z \,\middle|\, \begin{matrix} (-\mu,\delta),\ \{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},\ (-\rho-\mu-1,\delta) \end{matrix} \right], \qquad \ldots(1.6)$$

provided that Re $(\rho) > -1$, Re $(\mu + \delta\bar{\alpha}) > -1$, $|\arg z| < \frac{1}{2}\pi V$, where $\bar{\alpha}$ is given by (1.4).

$$\int_0^1 y^\rho (1-y)^\mu \; H[z(y-1)^\delta] \; J_\nu(xy^{1/k}) dy$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l (x/2)^{\nu+2l}}{l!\Gamma(l+\nu+1)} \; \Gamma\left(\rho+1+\frac{\nu+2l}{k}\right) H^{m,n+1}_{p+1,q+1}\left[(-1)^\delta z \,\middle|\, \begin{matrix} (-\mu,\delta),\{(a_p,\alpha_p)\} \\ \{(b_q,\beta_q)\},(-\rho-\mu-1-\frac{\nu+2l}{k},\delta) \end{matrix} \right], \qquad \ldots(1.7)$$

provided Re $(\rho+\nu/k) > -1$, Re $(\mu+1+\delta\bar{\alpha}) > 0$, $|\arg z| < \frac{1}{2}\pi V$, where $\bar{\alpha}$ and V are given by (1.4) and (1.3) respectively.

Remark 1 : Results (1.6) and (1.7) are direct consequences of the definition (1.1) and familiar integrals available in the literature.

Elastic deflection and bending stresses play an important and frequently even a primary role in handling of machinary systems, design of machine parts and recently in the field of air-craft structures. When lateral deflection exceeds one-half of the plate thickness, classical theory may not be applicable in general and the second order effects of the vertical displacements on the membrane stresses need to be taken into account. In this case, we are led to non-linear differential equation which can't be exactly solved.

Berger (1955) applied an approximate technique to large deflection problems which is essentially based upon neglecting the strain energy due to second invariant in the middle plane of the plate. The merit of the Berger's method lies in decoupling the non-linear equations, derived from the principle of virtual work, into a simple fourth order differential equation and a non-linear second order equation in such a way that one of them assumes a quasi-linear form and can be integrated fairly easily.

Notations

$$D = \frac{E h^3}{12(1-\theta^2)} = \text{flexural rigidity of the plate}$$

h = thickness of the plate
E = Yound's modulus
θ = Poisson's ratio
W= lateral displacement
u = radial displacement
$f(r)$ = non-uniform load
σ_r and σ_ϕ = bending stresses.

2. Statement of the Problem and Governing Equations

Let us consider a circular plate of radius a, thickness h and flexural rigidity D whose edge (i.e. the circumference) is claimed. We also assume that a non-uniform load $f(r)$ given by

$$f(r) = K_0\,[1-(r/a)^k]^\mu\, H^{m,n}_{p,q}\left[z\,\{(r/a)^k-1\}^s \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] \qquad \text{...(2.1)}$$

is imposed to the plate. If we neglect the strain energy due to second invariant in the middle plane of the plate, then following the Berger (1955) approximate method the equations for large deflection can be given as

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2W}{dr^2}\,\frac{1}{r}\frac{dW}{dr}-c^2W\right)=\frac{f(r)}{D}=F(r)\ (\text{say}) \qquad \text{...(2.2)}$$

where c is the normalized constant of integration given by

$$\frac{du}{dr}+\frac{u}{r}+\tfrac{1}{2}\left(\frac{dW}{dr}\right)^2=\frac{c^2h^2}{12}\,. \qquad \text{...(2.3)}$$

Since the circumference of the plate is clamped, the boundary conditions, therefore, for the problem are

$$W=0=\frac{dW}{dr}\ \text{when}\ r=a, \qquad \text{...(2.4)}$$

and $u=0$ when $r=a$. ...(2.5)

Remark 2 : Evidently the problem considered here is not without practical interest since we may have various non-uniform distribution of thrusts at points of contact when heavy bodies of different shapes are placed on the plate. Further motivation for our choice in (2.1) is that separate treatments for a number of cases may, due to general nature of the Fox's *H*-function, be avoided by this single treatment of the problem.

3. Solution

Let us suppose the solution of the problem in the form

$$W=\sum_i A_i\,[J_0(r\xi_i)-J_0(a\xi_i)], \qquad \text{...(3.1)}$$

ξ_i being the ith root of the equation

$$J_1(a\xi_i)=0. \qquad \text{...(3.2)}$$

Obviously, conditions (2.4) and (2.5) are satisfied by the solution assumed in (3.1). Combining (2.2) and (3.1) we arrive at

$$\sum_i A_i \xi_i^2 \left(c^2 + \xi_i^2\right) J_0\left(r\xi_i\right) = F(r). \qquad \ldots(3.3)$$

Now, multiply both sides of (3.3) by $r J_0(r\xi_i)$ and integrate with respect to r from 0 to a and use orthogonal property of Bessel functions Erdélyi (1953) to obtain

$$A_i = \frac{2}{a^2\xi_i^2 (c^2+\xi_i^2)[J_0(a\xi_i)]^2} \int_0^a rJ_0(r\xi_i)F(r)\,dr. \qquad \ldots(3.4)$$

A general solution to the problem may, therefore, be given as

$$W = \frac{2}{a^2} \sum_i \frac{[J_0(r\xi_i)-J_0(a\xi_i)]}{\xi_i^2(c^2+\xi_i^2)[J_0(a\xi_i)]^2} \int_0^a rJ_0(r\xi_i)F(r)\,dr, \qquad \ldots(3.5)$$

provided right-hand side of (3.5) makes sense.

Now, for our special choice for $F(r)$ as given in (2.2) and (2.1), appeal to the result (1.7) in (3.5) leads to

$$W = \frac{2K_0}{kD} \sum_i \frac{G(\xi_i)[J_0(r\xi_i)-J_0(a\xi_i)]}{\xi_i^2(c^2+\xi_i^2)[J_0(a\xi_i)]^2}, \qquad \ldots(3.6)$$

where
$$G(\xi_i) = \sum_{l=0}^{\infty} \frac{(-1)^l(a\xi_i/2)^{2l}}{l!\,\Gamma(l+1)} \frac{\Gamma\left(\frac{2l+2}{k}\right)}{} H^{m,n+1}_{p+1,q+1}$$
$$\times\left[(-1)^{\delta} z \left| \begin{matrix} (-\mu,\delta),\ \{(a_p,\alpha_p)\} \\ (b_q\beta_q),\ \left(-\frac{2}{k}-\frac{2l}{k}-\mu,\delta\right) \end{matrix} \right.\right], \qquad \ldots(3.7)$$

provided that the conditions given in (1.7) are satisfied. To determine the radial displacement u, introducing (3.1) into (2.3) and integrating with respect to r in view of the results Erdélyi (1953), we arrive at

$$ru = \frac{c^2h^2r^2}{24} - \tfrac{1}{2}\sum_{i=1}^{\infty} A_i^2\xi_i^2 \tfrac{1}{2} r^2\{[J_1(r\xi_i)]^2 - J_0(r\xi_i)J_2(r\xi_i)\}$$

$$-\tfrac{1}{2}\sum_{i=1}^{\infty}\sum_{\substack{j=1\\ i\neq j}}^{\infty} A_iA_j\xi_i\xi_j r(\xi_i^2-\xi_j^2)^{-1}\{\xi_iJ_2(r\xi_i)J_1(r\xi_j)-\xi_jJ_2(r\xi_j)J_1(r\xi_i)\}+C_1. \qquad \ldots(3.8)$$

The boundary condition (2.5) consequently demands

$$C_1 = -\frac{c^2h^2a^2}{24} + \tfrac{1}{4}\sum_{i=1}^{\infty} A_i^2\xi_i^2\, a^2[J_2(a\xi_i)]^2. \qquad \ldots(3.9)$$

If we let $c = 0$, the small deflection of the plate is given by

$$\overline{W} = \frac{2K_0}{kD} \sum_i \frac{G(\xi_i)[J_0(r\xi_i)-J_0(a\xi_i)]}{\xi_i^4[J_0(a\xi_i)]}. \qquad \ldots(3.10)$$

The bending stresses at the surface of the plate which, for the circular plate, are given by

$$\sigma_r = -\frac{6D}{h^2}\left(\frac{d^2W}{dr^2} + \frac{\theta}{r}\frac{dW}{dr}\right) \qquad \ldots(3.11)$$

and $$\sigma_\phi = -\frac{6D}{h^2}\left(\theta\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right) \qquad \ldots(3.12)$$

are found, for deflection given in (3.7), as

$$(\sigma_r)_{r=0} = (\sigma_\phi)_{r=0} = \frac{6K_0}{kh^2}\sum_i \frac{(1+\theta)G(\xi_i)}{(c^2+\xi_i^2)[J_0(a\xi_i)]^2} \qquad \ldots(3.13)$$

$$(\sigma_r)_{r=a} = \frac{12K_0}{kh^2}\sum_i \frac{G(\xi_i)}{(c^2+\xi_i^2)J_0(a\xi_i)} \qquad \ldots(3.14)$$

and $(\sigma_\phi)_{r=a} = \theta(\sigma_r)_{r=a}$. ...(3.15)

4. Particular Cases

(i) The large and small deflections at the centre of the plate may be deduced by letting $r = 0$ respectively in (3.7) and (3.11).

(ii) If the normal pressure is either of the forms

$$f(r) = K_0\,[1-(r/a)^k]^\mu \exp\{-z((r/a)^k-1)^\delta\}, \qquad \ldots(4.1)$$

$$f(r) = K_0\,[1-(r/a)^k]^\mu \sin\{2z^{1/2}((r/a)^k-1)^{\delta/2}\}, \delta>0 \qquad \ldots(4.2)$$

$$f(r) = K_0\,[1-(r/a)^k]^\mu \log\,[z^{1/2}((r/a)^k-1)+\sqrt{1+z((r/a)^k-1)^2}], \qquad \ldots(4.3)$$

etc. then large and small deflections (at an arbitrary point of the plate or at the centre) may be obtained from (3.7) and (3.10) with appeal to the known relationships (Mathai and Saxena 1978, p. 145).

5. Analysis of the Family of Load Shapes

If $m = 1 = z = \delta = \alpha_1 = \alpha_2 = \beta_1 = \beta_2$, $n = 2 = p = q$, $K_0 = \dfrac{'K_0}{\Gamma(b-\nu)}$, $a_1 = 1+\nu$, $a_2 = -\nu-\alpha-\beta$, $b_1 = 0$, $b_2 = -\alpha$, then within the limit when $b \to 0$, we have from (2.1)

$$f(r) = {}'K_0\,[1-(r/a)^k]^\mu\, \frac{\Gamma(-\nu+\alpha+\beta+1)}{\Gamma(1+\alpha)}\,{}_2F_1[-\nu,\nu+\alpha+\beta+1;\alpha+1;\,1-(r/a)^k]. \qquad \ldots(5.1)$$

Now, we shall discuss the following cases:

(i) If $\nu = 1$; $\alpha, \beta, \mu > 0$, then

$$f(r) = {}'K_0\,\frac{\Gamma(\alpha+\beta+2)}{\Gamma(1+\alpha)}\,[1-(r/a)^k]^\mu\left[1-\frac{\alpha+\beta+2}{1+\alpha}(1-(r/a)^k)\right] \qquad \ldots(5.2)$$

which shows that $f(r)=0$ when $r = a$ and (or) $r = a\left(\dfrac{\beta+1}{\alpha+\beta+2}\right)^{1/k}$. Also $f(r)$ is negative or positive according as $0 \leqslant r < a\left(\dfrac{\beta+1}{\alpha+\beta+2}\right)^{1/k}$ or $a\left(\dfrac{\beta+1}{\alpha+\beta+2}\right)^{1/k} < r < a$. We, therefore, conclude that the normal pressure over the plate area bounded by the circle $0 \leqslant r < a\left(\frac{\beta+1}{\alpha+\beta+2}\right)^{1/k}$ is acting in the negative direction of z-axis (upward) whereas in the annular region $a\left(\frac{\beta+1}{\alpha+\beta+2}\right)^{1/k} < r < a$, it is acting along positive z (downwards). We also observe that for fixed β, increases in the value of α increases the annular region whereas if α is kept constant, increase in β decreases the annular region. In otherwords, since the pressure distribution changes its sign, a bifurcation point is approached at $r = a\left(\frac{\beta+1}{\alpha+\beta+2}\right)^{1/k}$.

(ii) If we let $\nu = 0$, $\alpha = 0 = \beta$, $\mu > 0$, then

$$f(r) = 'K_0\,[1-(r/a)^k]^{\mu}, \qquad \ldots(5\text{-}3)$$

which represents an axially symmetric pressure distribution acting in the positive direction (downwards) if $'K_0 > 0$, of different intensities for different values of parameters k and μ.

(iii) Finally, if $\nu = 0$, $\alpha = 0 = \beta = \mu$, then

$$f(r) = 'K_0, \qquad \ldots(5.4)$$

which stands for a uniform force of magnitude $'K_0$ over the plate.

Acknowledgement

The authors wish to express their deep sense of gratitude to Prof. H. M. Srivastava (Victoria, Canada) for his invaluable discussions. Their sincere thanks are also due to the referee for useful suggestions.

References

Berger, H. M. (1950). A new approach to the analysis of large deflection of plates. *J. appl. Mech. Trans. ASME*, 22, 465–72.

Braaksma, B. L. J. (1964). Asymptotic expansion and analytic continuations for a class of Barnes-integrals. *Compositio Math.*, **15**, 239–341.

Erdélyi, A. (1953). Higher Transcendental Functions, Vol. II. McGraw-Hill Book Co., Inc., New York.

Fox, H. (1961). The *G* and *H* functions as symmetrical Fourier kernels. *Trans. Am. Math. Soc.*, 98, 395–429.

Mathai, A. M., and Saxena, R. K. (1978). The *H*-function with Applications in Statistics and Other Disciplines. Wiley Eastern Ltd., New Delhi.