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POWER SERIES SOLUTIONS OF COMPOSITE POLYTROPIC MODELS

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In this paper it is shown that it is possible to represent the solutions to the differential equations governing the structure of a composite polytropic model in terms of convergent power series.

INTRODUCTION

In an earlier paper (Mohan and Al-Bayaty 1980) we have shown that it is possible to construct power series solutions of the Lane-Emden equation which are convergent in the whole interior of a polytropic model for values of the polytropic index n between 0 and 5. In the present paper we consider the problem of representing the interiors of composite polytropic models by means of convergent power series solutions. By a composite polytropic model we mean a spherical model in which the polytropic index n has different values in the different concentric spherical regions of the model. Composite polytropic models are sometimes used in literature to approximate the internal structure of a star in which the equilibrium conditions differ from one spherical zone to another. Stars with convective cores and radiative envelopes or vice versa and stars in the later stages of evolution are some of the examples of the stars in which equilibrium conditions differ from one concentric spherical zone to another (cf. Cox and Giuli 1968). Unfortunately in the case of composite polytropic models it is not possible to obtain the solutions to the differential equations representing their structure in closed analytic form. Therefore, in the absence of solutions in closed form, convergent power series solutions to these differential equations can serve as analytic representation of the interiors of composite polytropic models. In this connection it may be mentioned that the power series is one of the powerful methods of mathematical analysis and is no less (and sometimes even more) convenient than the elementary functions especially when the problems are to be studied on computers. (In fact the modern computers often use series in the calculation of the majority of the elementary functions). There are also other practical advantages of obtaining convergent power series solutions to a composite polytropic model for which analytic solutions are not possible. Since a convergent power series can be integrated or differentiated term by term inside its circle of convergence, we can conveniently use these convergent series in the theoretical studies involving differentiation or integration of the unknown function.

In section 2 we discuss the mathematical structure of a composite polytropic model. In section 3 we explain the method for obtaining convergent power series solutions to a composite polytropic model. The problem of constructing a desired composite polytropic model by the use of convergent power series solutions is also discussed in this section and illustrated through certain numerical examples. In section 4 we present certain concluding remarks based on our present study.

2. A Composite Polytropic Model

A polytropic model of index n is a gas sphere in which the density ρ and the pressure p at a point inside the sphere are given by the relations

$$\rho = \rho_c\theta^n \text{ and } p = p_c\theta^{n+1} \quad \text{...(1)}$$

where ρ_c and p_c are the values of ρ and p at the centre and θ is some unknown variable depending upon the distance of the point from the centre. In polytropic models of physical interest which have relevance to the models of actual stars, n lies between 0 and 5. Theoretically a polytropic model of index n is a solution of the second order nonlinear differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \quad \text{...(2)}$$

with the initial conditions

$$\theta = 1, \; \frac{d\theta}{d\xi} = 0 \quad \text{at } \xi = 0. \quad \text{...(3)}$$

Equation (2) is the well known Lane-Emden equation (LEE for brevity) of index n. In eqn. (2) ξ is the independent variable which is related to the distance r of an element of the gas sphere from the centre of the sphere by the relation (cf. Chandrasekhar 1957, p. 87)

$$r = \left[\frac{(n+1)p_c}{4\pi G\,\rho_c^2}\right]^{1/2} \cdot \xi\,. \quad \text{.. (4)}$$

G being the universal gravitational constant. Also $M(r)$ the mass of thc fluid contained inside a sphere of radius r is given by the relation

$$M(\xi) = -4\pi\left[\frac{(n+1)p_c}{4\pi G\rho_c^2}\right]^{3/2} \cdot \rho_c\left(\xi^2\frac{d\theta}{d\xi}\right). \quad \text{.. (5)}$$

The free surface of the gas sphere which has polytropic structure of index n is given by (4) for the value of ξ at which the first zero of θ occurs in the solution of the LEE (2). In general it is not possible to obtain analytic solutions to the LEE (2) for values of n between 0 and 5 except for $n=0$, 1 and 5.

A composite polytropic model consists of at least two concentric spherical zones, one say core satisfying the LEE (2) for one value of n (say n_1) surrounded by a spherical envelope satisfying the LEE (2) for another value of n (say n_2). A composite polytropic model can have more than two zones also. The initial conditions at the centre of a composite model are still given by (3). Therefore, in a composite polytropic model consisting of two zones the structure of the core is still described by the solution of (2) subject to (3) for $n=n_1$. However, now the boundary conditions for the solution of the envelope are determined by the requirement that the density and pressure be continuous across the interface between the envelope and the core. Therefore in a composite polytropic model consisting of two zones, the envelope is not exactly a solution of the LEE for $n=n_2$ but is one of the several solutions which are determined from the LEE by use of the homology theorem. According to the homology theorem (cf. Chandrasekhar 1957, p. 101), 'If $\theta(\xi)$ is a solution of the LEE (2) then for every arbitrary real value of A, $A^{2(n-1)}\,\theta(A\xi)$ is also a solution of the LEE (2)'.

In order to avoid confusion we shall restrict the use of variables θ and ξ to the core of polytropic index n_1 and instead use variables ϕ and η for the envelope of polytropic index n_2. To maintain the continuity of density and pressure across the interface between the envelope and the core we may define the density and pressure in the envelope to be given by the relations

$$\rho=\rho_i\left(\frac{\phi}{\phi_i}\right)^{n_2},\ p=p_i\left(\frac{\phi}{\phi_i}\right)^{n_2+1} \qquad ...(6)$$

where subscript i is used to denote the value of a variable at the interface between the envelope and the core. Also from the core solutions we obtain at the interface

$$\rho_i=\rho_c\,\theta_i^{\,n_1},\ p_i=p_c\,\theta_i^{\,n_1+1} \qquad ...(7)$$

Combining (6) and (7) we can write

$$\rho=\rho_c\,\frac{\theta_i^{n_1}}{\phi_i^{n_2}}\,\phi,\ p=p_c\,\frac{\theta_i^{n_1+1}}{\phi_i^{n_2+1}}\,\phi \qquad ...(8)$$

in the envelope. Using (4), (5) and (8) we can write the values of r and $M(r)$ at the interface from the core solution and the envelope solution. On equating the values of r at the interface as obtained from the core and envelope solutions we get

$$\left[\frac{(n_1+1)p_c}{4\pi G\rho_c^2}\right]^{1/2}.\,\xi_i=\left[\frac{(n_2+1)p_c}{4\pi G\rho_c^2}\,\frac{\phi_i^{n_2-1}}{\theta_i^{n_1-1}}\right]^{1/2}.\eta_i\,. \qquad ..(9)$$

Similarly on equating the values of $M(r)$ at the interface as obtained from the core and envelope solutions we get

$$\left[\frac{(n_1+1)p_c}{4\pi G\rho_c^2}\right]^{3/2}\rho_c\left(\xi^2\,\frac{d\theta}{d\xi}\right)_i=\left[\frac{(n_2+1)p_c}{4\pi G\rho_c^2}\,\frac{\phi_i^{n_2-1}}{\theta_i^{n_1-1}}\right]^{3/2}\rho_c\times\frac{\theta_i^{n_1}}{\phi_i^{n_2}}\left(\eta^2\,\frac{d\phi}{d\eta}\right)_i. \quad ...(10)$$

Raising (9) to the third power and dividing by (10) we get

$$\left(\xi\theta^{n_1}/\frac{d\theta}{d\xi}\right)_i=\left(\eta\phi^{n_2}/\frac{d\phi}{d\eta}\right)_i\,. \qquad ...(11)$$

Also dividing (10) by (9) we get

$$\left(n_1+1\right)\left(\xi\frac{d\theta}{d\xi}/\theta\right)_i=(n_2+1)\left(\eta\frac{d\phi}{d\eta}/\phi\right)_i. \qquad ...(12)$$

Equations (11) and (12) can be used as equations of fit to determine a composite polytropic model of index n_1 in the core and index n_2 in the envelope with interface at a desired point. If a composite model has more than two zones then conditions (11) and (12) will have to be satisfied at each interface.

In order to construct a composite polytropic model consisting of two zones, first the solution of the LEE of index n_1 is obtained for the core. Then a family of homology solutions of the LEE of index n_2 is constructed. Finally equations (11) and (12) are used to determine the value of the homology constant A for which these equations are satisfied at the desired point of the interface. In the same way a composite polytropic model with more than two zones can also be constructed.

3. Representation of a Composite Polytropic Model by Convergent Power Series

We have shown in our earlier paper (Mohan and Al-Bayaty 1980) that (i) the power series solution to the Lane-Emden equation of index n developed at an arbitrary point x_0 inside the polytropic sphere where

$$x_0=\xi_0/\xi_1 \qquad (0<x_0<1)$$

ξ_0 being the value of ξ at the chosen point and ξ_1 the value of ξ where the first zero of the Lane-Emden equation of the given index n occurs, in terms of the non-dimensional variable $x=\xi/\xi_1$ is given by

$$\theta=\theta_0+\sum_{k=1}^{\infty}a_k(x-x_0)^k \qquad ...(13)$$

where

$$a_1=\theta_0'$$

$$a_2=-\tfrac{1}{2}(x_0\xi_1^2\ \theta_0^n+2\theta_0')$$

$$a_3=-\frac{1}{6}\left[(1+x_0c_1)\xi_1^2\ \theta_0^n+6a_2\right]$$

and in general for $k\geqslant1$

$$a_{k+2}=-\left[(c_{k-1}+x_0c_k)\xi_1^2\ \theta_0^n+(k+1)(k+2)a_{k+1}\right]\Big/\left[(k+1)(k+2)\right]$$

with $c_0=1$

$$\text{and}\quad c_k=\frac{1}{k\theta_0}\sum_{i=1}^{k}(in-k+i)a_ic_{k-i}\ ,\ k\geqslant1$$

θ_0 and θ_0' being the values of θ and $d\theta/dx$ at x_0.

(ii) The power series solution to the Lane-Emden equation developed at the centre $x_0=0$ is given by

$$\theta=1+\sum_{k=1}^{\infty} a_k x^{2k} \qquad ...(14)$$

where

$$a_1=-\frac{1}{6}\xi_1^2$$

$$a_2=\frac{n}{120}\xi_1^4$$

$$a_3=-\frac{n(8n-5)}{3.7!}\xi_1^6$$

and in general

$$a_{k+1}=\frac{1}{k(k+1)(2k+3)}\sum_{i=1}^{k}(in+i-k)(k-i+1)\times(3+2k-2i)a_i a_{k-i+1},\ k\geqslant 1.$$

(iii) The power series solution to the LEE developed at the surface $x=1$ (valid only for integral values of n other than one) is given by

$$\theta=\sum_{k=1}^{\infty} a_{k-1}(1-x)^k \qquad ...(15)$$

where

$$a_0=-\theta_0'$$
$$a_k=a_0 \qquad k\leqslant n$$
$$a_{n+1}=a_0-\xi_1^2\ b_0^n\Big/\Big[(n+1)(n+2)\Big]$$

and $a_{k+1}=a_k-\xi_1^2\ a_0^n(c_{k-n}-c_{k-n-1})/[(k+1)(k+2)] \quad k>n$

$$\text{with } c_k=\frac{1}{a_0 k}\sum_{i=1}^{k}(in-k+i)a_i c_{k-1} \text{ and } c_0=1.$$

For convenience, the numerical values of the input parameters θ_0 and θ_0' to be used in the power series solutions (13) developed at $x_0=0.2$, 0.5 and 0.7 and for power series solutious (14) and (15) are listed in Table I for different values of the polytropic index n. The radii of convergence of these power series solutions are also being presented in Table II for reference.

Now suppose we have a composite polytropic model of index n_1 in the core and index n_2 in the envelope for which the position of the interface as well as the value of the homology constant A are known and it is desired to represent its interior in terms of convergent power series solutions. Then series (14) can be used with n

TABLE I

Numerical values of the input parameters θ_0 and θ_0' to be used in a series solution developed at x_0

x_0	θ_0	θ_0'	x_0	θ_0	θ_0'
	$n=0.5,\ \xi_1=2.7528$			$n=1.0,\ \xi_1=3.14159$	
0.2	0.9498632	−0.4975089	0.2	0.9354885	−0.632360
0.5	0.6994500	−1.140322	0.5	0.6986449	−1.204912
0.7	0.4406113	−1.421260	0.7	0.3678811	−1.365235
1.0			1.0	0.0	−1.00
	$n=1.5,\ \xi_1=3.65375$			$n=2.0,\ \xi_1=4.35287$	
0.2	0.9144607	−0.8217741	0.2	0.8826591	−1.089249
0.5	0.6286134	−1.346670	0.5	0.5433201	−1.428216
0.7	0.2963826	−1.226457	0.7	0.2306240	−1.026561
1.0			1.0	0.0	−0.5538989
	$n=2.5,\ \xi_1=5.35528$			$n=3.0,\ \xi_1=6.89685$	
0.2	0.8329170	−1.458852	0.2	0.7532146	−1.921322
0.5	0.3784914	−1.269791	0.5	0.2840250	−1.049560
0.7	0.1733420	−0.8037008	0.7	0.1250870	−0.5907226
1.0			1·0	0.0	−0.2926321
	$n=3.5,\ \xi_1=9.53581$			$n_1=4,\ \xi_1=14.97155$	
0.2	0.6265230	−2.324831	0.2	0.4409082	−2.235185
0.5	0.1965300	−0.7654043	0.5	0.1198634	−0.4766314
0.7	0.0849627	−0.4036343	0.7	0.05146220	−0.2449046
1.0			1.0	0.0	−0.1200430
	$n=4.5,\ \xi_1=31.83646$			$n_1=4.9,\ \xi_1=169.47$	
0.2	0.2159184	−1.298751	0.2	0.04087132	−0.2540918
0.5	0.05498982	−0.2182217	0.5	0.01035999	−0.04069394
0.7	0.02380169	−0.1114102	0.7	0.004546478	−0.02076241

replaced by n_1 to represent the structure of the core provided this series is convergent in the entire core. Otherwise we can use (13) to develop power series solution for the core choosing x_0 in such a way that the solution is convergent in the entire core region. Power series solution (15) can be used to develop the power series solution $\phi(\eta)=A^{2/(n_2-1)}\,\theta(A\xi)$ for the envelope provided n_2 is an integer other than one (for $n=1$, analytic solutions exist) and the solution is convergent in the entire region of the envelope, otherwise this series solution can be generated with the help of (13) choosing x_0 in such a way that the solution is convergent in the entire region of the envelope. For this purpose values of the radii of convergence of the various power series solutions given in Table II can serve as guidelines. The same method

TABLE II

Radii of convergence of the power series solution developed at the point x_0 for different values of n, taking the radius of the model as unity.

Values of the radii convergence

$x_0 \setminus n$	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	4.9
0.0	1.0	1.0	1.0	0.8	0.4	0.35	0.2	0.1	0.04	0.00
0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.15	0.15
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.7	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
1.0	—	1.0	—	1.0	—	0.9	—	0.9	—	—

can be used to construct convergent power series solutions to models with more than two zones.

We can also use power series (13), (14) and (15) to construct a composite model with interface at a desired point. For this purpose we select a power series from (13) or (14) for the polytropic index n_1 of the core ensuring that it is convergent in the entire region of the core. We now use this series to determine the values of θ, θ^{n_1}, $d\theta/d\xi$ at various points in the core. (Power series for θ being convergent it can be differentiated term by term to get series expansion for $d\theta/d\xi$. The value of θ^{n_1} can be calculated by raising the computed values of θ to n_1). Similarly we use (14) or (15) to select a convergent power series solutions for the value of polytropic index n_2 of the envelope and then use this series solution and the homology theorem to write the values of η, ϕ, ϕ^{n_1}, $d\phi/d\eta$ at various points of the envelope for different choices of A. The two sets of solutions are then compared with the help of eqns. (11) and (12) to determine the values of A and η for which the solutions match at the interface.

We have used the power series solutions technique explained above to reconstruct some of the composite polytropic models which were earlier constructed by Singh (1967) using numerical integration techniques. The results are presented in Table III. For comparison we also give in this table the values of A for these models as were determined by Singh. In Table IV we present the values of dependent

TABLE III

Comparison between the values of homology constant A for certain composite polytropic models as found by the power series solution method with the corresponding values of A as found by the numerical integration technique

Model	n_1	n_2	x_i	A	
				By series solution	By numerical integration
I	1.5	3.0	0.2094	7.142795	7.142857
II	1.5	3.0	0.6160	4.482352	4.482124
III	1.5	3.0	0.8250	3.939374	3.939394

* Results taken from Singh (1967).

TABLE IV

Internal structure of a composite polytropic model with $n_1=1.5$, $n_2=3.0$, with interface at 0.6160 as determined from the power series solutions

Core		Envelope	
x	θ	x	ϕ
0.0	1.0	0.6160	2.43572
0.10	0.96702	0.65	2.12802
0.20	0.87379	0.70	1.71185
0.30	0.73574	0.75	1.33917
0.40	0.57339	0.80	1.00727
0.50	0.40715	0.85	0.71189
0.60	0.25310	0.90	0.44841
0.6160	0.23033	1.00	0.00000

variable θ in the core and ϕ in the envelope for the composite model with $n_1=1.5$ in the core and $n_2 = 3$ in the envelope with interface at $x=.616$. These values were found from the power series solutions for the core and the envelope of this model and agree with the corresponding results found by numerical integration upto at least four significant figures in each case. (In the actual calculations the series summation process was carried upto a term after which the succeeding terms of the series did not make any contribution to the first six significant figures. In each case the number of terms needed for this purpose was less than twenty).

4. CONCLUDING REMARKS

In the present work we have shown how the interiors of the composite polytropic models can be expressed in terms of convergent power series. We have also shown that it is even possible to construct a composite polytropic model using power series solutions. The method of constructing composite models with the help of power series solutions is in no way more laborious than the usual method of numerical integration and is of comparable accuracy. However, the added advantage of this method is that we get convergent power series solutions for the unknown variables θ, θ' and ϕ, ϕ' of the core and envelope as a by-product. As mentioned earlier the main advantage of representing the interior of a composite polytropic model in terms of convergent power series is that in the absence of analytic solutions in closed form, these convergent power series can be used as the analytic representation of the solutions and can be used in further analytic studies concerning these models. While writing the power series solutions to a composite polytropic model care must be taken that the power series developed for a particular region is convergent in that entire region. For this purpose the values of the radii of convergence given in Table II can serve as guidelines. If necessary one can even use (13) to develop a power series at a point x_0 other than 0.2, 0.5, 0.7 to ensure its convergence in a particular region.

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