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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE CLASS LIP (α, p) BY MEANS OF A CONJUGATE SERIES

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In this paper the author has determined the degree of approximation of certain functions belonging to the class Lip (α, p) by Nörlund means. The result obtained generalizes his previous result proved in Theorem A (Qureshi 1981).

§ 1. Let $f(x)$ be a function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let its Fourier series be given by

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x) \qquad \ldots(1.1)$$

then

$$\bar{f}(x) \sim \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \qquad \ldots(1.2)$$

is called the conjugate series of $f(x)$.

We shall use the notation $\psi(t) = f(x+t) - f(x-t)$. $\qquad \ldots(1.3)$

We define the norm $\| \cdot \|_p$ as

$$\| f \|_p = \left\{ \int_0^{2\pi} | f(x) |^p \, dx \right\}^{1/p}, \; p \geqslant 1, \qquad \ldots(1.4)$$

and let the degree of approximation $E_n(f)$ be given by (see Zygmund 1959)

$$E_n(f) = \min_{T_n} \| f - T_n \|_p \qquad \ldots(1.5)$$

where $T_n(x)$ is a trigonometrical polynomial of degree n. We say that

$f(x) \in$ Lip (α, q) for $a \leqslant x \leqslant b$

if

$$\{ \int_a^b | f(x+h) - f(x) | \leqslant A | h |^{\alpha}, 0 < \alpha \leqslant 1 \qquad . \; (1.6)$$

[see Def. 5.38 of McFadden (1942)].

§ 2. Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the (N, p_n) method such that

$$P_n \equiv P(n) = p_0 + p_1 + \text{——} + P_n \to \infty \text{ as } n \to \infty. \qquad \ldots(2.1)$$

We write $p(y) = p_{[y]}$ and $P(y) = P_{[y]}$,

where $[y]$ as usual denotes the greatest integer less than y.

We have proved the following theorem (see Qureshi 1981).

Theorem A—If the sequence $\{p_n\}$ satisfies the following conditions

$$n \, | p_n | < C \, | P_n |$$

and $\sum_{k=1}^{n} k \mid p_k - p_{k-1} \mid < C \mid P_n \mid$

then the degree of approximation of $f(x)$, conjugate to a periodic function f with period 2π and belonging to the class of Lip α, $0 < \alpha \leqslant 1$, λ by (N, p_n) means of its conjugate series, is given by

$$\left| \bar{f}(x) - \bar{t}_n(x) \right| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}} \right),$$

where $\bar{t}_n(x)$ are the (N, p_n)—means of the conjugate series (1.2).

Our object here is to prove the following theorem:

Theorem B—If $f(x)$ is periodic and belongs to the class Lip (α, p) for $0 < \alpha \leqslant 1$, and if the sequence $\{p_n\}$ is as defined in (1.7) with the other requirements there in and if

$$\left(\int_1^n \frac{P(y)^q}{y^{q\alpha+2+\delta q-2}} \right)^{1/q} = O\left(\frac{P(n)}{n^{\alpha+(1/q)+\delta-1}} \right)$$

then $\| \bar{f} - \bar{t}_n \|_p = O\left(\frac{1}{n^{\alpha-(1/p)}} \right)$.

where $\bar{t}_n(x)$ is the (N, p_n) mean of the series (1.2),

$\frac{1}{p} + \frac{1}{q} = 1$ such that $1 \leqslant p \leqslant \infty$ and δ is an arbitarary poistive number such that $q(1-\delta) - 1 > 0$.

Following lemmas are known :

Lemma 1 (Sahney and Goel 1973, Lemma 1)—If the sequence $\{p_n\}$ is positive and non-increasing, then, for $\alpha > 0$

$$\frac{1}{n^\alpha} \leqslant \frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}.$$

Lemma 2 (McFadden 1942, Lemmas 5.11)—If $\{p_n\}$ is non-negative and non-increasing then for $0 \leqslant a \leqslant b \propto$; $0 \leqslant t \leqslant \pi$ and any n, we have

$$\left| \sum_{k=a}^{q} p_k \, e^{(n-k)t} \right| \leqslant P\left(\frac{1}{t} \right) \text{ for any } a.$$

Lemma 3 (McFadden 1942, Lemma 5.40)—If $f(x)$ belongs to Lip (α, q) on $[0, \pi]$, then $\psi(t)$ also belongs to Lip (α, q) on $[0, \pi]$.

§ 3. *Proof of the Theorem*—After Qureshi (1981) we write

$$\bar{f}(x) - \bar{t}_n(x) = \frac{1}{\pi} \int_0^\pi \psi(t) \; \frac{1}{P_n} \sum_{k=9}^{n} p_{n-k} \frac{\cos(k+\frac{1}{2})t}{2 \sin t/_2} \, dt$$

where $\psi(t) = f(x+t) - f(x-t)$.

We have $\bar{f}(x) - \bar{t}_n(x) = \frac{1}{2\pi P_n} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right) \frac{\psi(t)}{\sin t/2} \sum_{k=0}^{n} p_{n-k} \cos (k+\frac{1}{2})\, t\, dt$

$= I_1 + I_2$, say,

Now $I_1 = \frac{1}{2\pi P_n} \int_0^{\pi/n} \frac{\psi(t)}{\sin t/2} \sum_{k=0}^{n} p_{n-k} \cos (k+\frac{1}{2})\, t\, dt.$

By Hölders inequality and Lemma 3, we have

$$I_1 \leqslant \frac{1}{2\pi P_n} \left(\int_0^{\pi/n} \left(\frac{t\,|\psi(t)|}{t^{\alpha}} \right)^p dt \right)^{1/p}$$

$$\left(\int_0^{\pi/n} \left(\frac{1}{\sin (t/2)\, t^{1-\alpha}} \left| \sum_{k=0}^{n} p_{n-k} \cos (k+\tfrac{1}{2})\, t \right| \right)^q dt \right)^{1/q}$$

$$= O\left(\frac{1}{P_n} \right) O\left(\frac{1}{n} \right) O\left\{ \left(\int_0^{\pi/n} \left(\frac{1}{t^{2-\alpha}} \sum_{h=0}^{n} p_{n-k} \right)^q dt \right)^{1/q} \right\}$$

$$= O\left(\frac{1}{n} \right) O\left\{ \left(\int_0^{\pi/n} t^{\alpha q - 2q}\, dt \right)^{1/q} \right\}$$

$$= O\left(\frac{1}{n} \right) O\left\{ \left(\frac{1}{n} \right)^{\alpha - 2 + 1/q} \right\} = O\left\{ \left(\frac{1}{n} \right)^{\alpha - 1/p} \right\}$$

Also $I_2 = \frac{1}{2\pi P_{nq}} \int_{\pi/n}^{\pi} \frac{\psi(t)}{\sin(t/2)} \sum_{k=0}^{n} p_{n-k} \cos (k+\frac{1}{2})\, t\, dt.$

Similarly, as above, we have

$$I_2 = O\left(\frac{1}{P_n} \right) \left\{ \left(\int_{\pi/n}^{\pi} \left(t^{-\delta} \frac{|\psi(t)|}{t^{\alpha}} \right)^p dt \right)^{1/p} \right\} \cdot$$

$$\left\{ \left(\int_{\pi/n}^{\pi} \left| \frac{P\left(\frac{1}{t}\right)}{t^{1-\delta-\alpha}} \right|^q dt \right)^{1/q} \right\} \text{ (by lemma 2)}$$

$$= O\left(\frac{1}{P(n)} \right) \left\{ \left(\int_{\pi/n}^{\pi} \left(t^{-\delta} \frac{t^{\alpha-(1/p)}}{t^{\alpha}}\, dt \right)^{1/p} \right\} \left\{ \left(\int_1^{n} \left(\frac{P(y)}{y^{\alpha+\delta-1}} \right)^q \frac{dy}{y^2} \right)^{1/q} \right\}$$

$$= O\left(\frac{1}{P(n)} \right) \left\{ \left(\int_{\pi/n}^{\pi} t^{-\delta p - 1}\, dt \right)^{1/p} \right\} \left\{ \left(\int_1^{n} \frac{(P\, y)^q}{y^{q\alpha + q\delta - q + 2}}\, dy \right)^{1/q} \right\}$$

(*equation Contd. on p.* 563)

$$= O\left(\frac{1}{P(n)}\right) O\left\{\left(\frac{1}{n}\right)^{-\delta}\right\}\left\{O\left(\frac{P(n)}{n^{\alpha+\delta-1+(1/q)}}\right)\right\}$$

$$= O\left\{\left(\frac{1}{n}\right)^{\alpha-1+1/q}\right\} = O\left\{\left(\frac{1}{n}\right)^{\alpha-1/p}\right\}.$$

Hence $\left|\bar{t}_n(x) - \bar{f}(x)\right| = O\left\{\left(\frac{1}{n}\right)^{\alpha-(1/p)}\right\}.$

Therefore $\|\bar{f}(x) - \bar{t}_n(x)\|_q = O\left[\left\{\int_0^{2\pi}\left(\left(\frac{1}{n}\right)^{\alpha-(1/p)}\right)^p dx\right\}^{1/p}\right]$

$$= O\left[\left(\frac{1}{n}\right)^{\alpha-(1/p)}\left(\int_0^{2\pi} dx\right)^{1/p}\right] = O\left[\left(\frac{1}{n}\right)^{\alpha-(1/p)}\right].$$

This completes the proof of the theorem.

Remark : It is to be noted with the help of Lemma 1 that as $p \to \infty$ (and therefore $q = 1$), our Theorem B is equivalent to Theorem A.

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