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NECESSARY AND SUFFICIENT CONDITIONS FOR THE ABSOLUTE EULER SUMMABILITY

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In this paper the author has investigated necessary and sufficient conditions for the absolute Euler summability of Fourier series and allied series. He has further shown that the sequence of factors involved in allied series cannot be improved.

1. DEFINITIONS AND NOTATIONS

Let $\Sigma\, d_n$ be a given infinite series, where summation is over $n = 0, 1, \ldots$, and let q be a real or complex number such that $q \neq -1$. Then, we write

$$d_n^q = (1+q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m}\, d_m \;;\quad d_n^0 = d_n.$$

Following the earlier work (Chandra 1975), we write

$$\sum d_n \in |E, q| \Longleftrightarrow \sum |d_n^q| < \infty .$$

Let f be a real, 2π-periodic function and L-integrable over $[-\pi, \pi]$ and let its Fourier series, at a point x, be

$$\tfrac{1}{2}\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2}\, a_0 + \sum_{n=1}^{\infty} A_n(x).$$

Throughout the paper, we assume $a_0 = 0$ and write for fixed real number x,

$$2\,\phi(t) = f(x+t) + f(x-t)$$

$$t\,\phi_1(t) = \int_0^t \phi(u)\, du$$

$$P(t) = \phi(t) - \phi_1(t)\, .$$

We also write T for the integral part of $(k/t)^2$ where $k > \pi e^2$.

2. INTRODUCTION

Mohanty and Mohapatra (1968) gave the following criterion for the absolute Euler summability of a Fourier series :

Theorem A—Let

$$\phi(t) \log (k/t) \in BV[0, \beta],\ 0 < \beta < 1. \qquad \ldots(2.1)$$

Then

$$\Sigma\, A_n(x) \in |E, q|\ (q > 0). \qquad \ldots(2.2)$$

The present author (Chandra 1972 a, b) obtained the above theorem with (2.1) replaced by

$$P(t)\left(\log\frac{k}{t}\right)^{1+\epsilon} \in BV[0,\beta],\ 0<\beta<1,\ \epsilon>0. \qquad \ldots(2.3)$$

It has been shown, recently, by the present author (Chandra 1980) that the condition (2.3) is best possible in the sense that "ϵ" cannot be dropt.

In view of the above fact, we propose to investigate the following set of conditions for the generating function of Fourier series to ensure (2.2) :

$$(a)\quad P(0+)=0 \text{ and } (b)\ \int_0^\beta \log(k/t)\,|dP(t)| < \infty. \qquad \ldots(2.4)$$

It is clear that (2.4), which is equivalent to

$$\text{(i) } P(t)/t \in L\,[0,\beta] \text{ and (ii) } P(t)\log(k/t) \in BV\,[0,\beta], \qquad \ldots(2.5)$$

is a weaker condition than (2.3). Also it may be observed that the condition (2.4) is also weaker than (2.1). Thus, our first attempt is to prove the following theorem which improves the above mentioned results :

√*Theorem* 1—Let $0<\beta<1$ and let (2.4b) hold. Then, in order that (2.2) should hold, it is necessary and sufficient that (2.4a) holds.

Now, we further remark that the condition (2.4b) of Theorem 1 is best possible in the sense that it cannot be replaced by the weaker condition:

$$\int_0^\beta g(t)\log(k/t)\,|dP(t)| < \infty,\ 0<\beta<1, \qquad \ldots(2.6)$$

where positive function g, defined on $[0,\beta]$, is such that

$$g(t) = o(1) \text{ as } t\to 0+. \qquad \ldots(2.7)$$

A proof of this remark is contained in Chandra and Dixit (1981).

Regarding the absolute Euler summability factor for the Fourier series, Tripathi (1974) has recently proved the following :

Theorem B—Let $y_n = (\log(n+1))^{-1}$ and let

$$\phi(t) \in BV[0,\pi]. \qquad \ldots(2.8)$$

Then

$$\sum_{n=1}^{\infty} A_n(x)\,y_n \in |E,q|\ (q>0). \qquad \ldots(2.9)$$

Replacing (2.8) by (2.4a) and

$$\int_0^\pi \log\log(k/t)\,|dP(t)| < \infty \qquad \ldots(2.10)$$

in Theorem B, we prove the following :

Theorem 2—Let (2.10) hold. Then in order that (2.9) should hold it is necessary and sufficient that (2.4a) should hold. Further the factor (y_n) cannot be replaced by (y_n^δ), where $0<\delta<1$, and also the result becomes false if we replace (2.10) by

$$\int_0^\pi (\log\log(k/t))^\delta\,|dP(t)| < \infty,\ \text{where } 0<\delta<1. \qquad \ldots(2.11)$$

It may be observed that the functions satisfying (2.10) and (2.4a) do not, in general, satisfy (2.8). In fact, even unbounded functions satisfy the conditions (2.10) and (2.4a).

We note that either (2.4b) or (2.10) implies that $P(t)$ tends to some limit as $t \longrightarrow 0+$. If (2.4a) is false, then there is no real loss of generality in supposing that this limit is positive (since otherwise we may apply the result in this case with $f(t)$ replaced by $-f(t)$). Thus the necessity part of Theorems 1, 2 is included in the following more general theorem:

Theorem 3—Suppose that $P(t)$ tends to a positive limit as $t \longrightarrow 0+$. Then the series

$$\sum_{n=1}^{\infty} A_n(x) \quad \text{and} \quad \sum_{n=1}^{\infty} A_n(x)\, y_n$$

diverge to $-\infty$; thus they are not summable (and, *a fortiori*, not absolutely summable) by any totally regular method.

3. Lemmas

We shall use the following lemmas in the proof of the theorems :

Lemma 1—Let $0 < \beta < 1$ and $F(t) \in L[\beta, \pi]$. Then $\sum_{n=1}^{\infty} \alpha_n \in |E, q|$ $(q>0)$,

where $\alpha_n = \int_{\beta}^{\pi} F(t) \exp(int)\, dt.$

Lemma 2—Let $q > 0$. Then, uniformly in $0 < t \leqslant \beta < 1$,

$$\sum_{n=0}^{\infty} (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\exp(imt)}{m+1} \right| = O\{\log(1/t)\}.$$

Lemma 3—Let $q > 0$. Then, uniformly in $0 < t \leqslant \pi$,

$$\sum_{m=1}^{n} (1+q)^{-n-1} \binom{n+1}{m+1} q^{n-m} \frac{\exp(imt)}{\log(m+2)} = O\left\{ t^{-1} \left(\log \frac{2\pi}{t}\right)^{-1} n^{-\frac{1}{2}} \right\}.$$

For Lemmas 1 and 2, reference may be made to Mohanty and Mohapatra (1968) and Chandra (1978a) respectively while, for Lemma 3, a reference may be made to Chandra (1978b).

Lemma 4—The $|E, q|$ method is translative for $q > 0$.

A proof of this lemma is contained in Agnew (1944).

To ensure that it is possible to choose $\phi(t)$ so that $P(t)$ has a given value, we give the following lemma.

Lemma 5—Let $Q(t) \in L[0, \pi]$. If we define

$$\phi(t) = Q(t) - \int_{t}^{\pi} (Q(u)/u)\, du. \qquad \ldots(3.1)$$

then $\phi(t) \in L[0, \pi]$; and $P(t) = Q(t)$.

PROOF : The verification that $\phi(t) \in L\,[0, \pi]$ is straightforward. If we substitute (3.1) in the definition of $P(t)$ we find, after some straightforward manipulation, that $P(t)$ reduces to $Q(t)$.

4. PROOF OF THE THEOREMS

Proof of Theorem 1—We have, by the proof of the present author (Chandra 1978a) ; Theorem 1,

$$A_n(x) = \frac{2}{\pi}\int_{\beta}^{\pi} \phi(t)\cos nt\,dt + \frac{2}{\pi}\,\phi_1(\beta)\,\frac{\sin n\beta}{n}$$

$$+\frac{2}{\pi}\int_0^{\beta} t\,P(t)\,\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt$$

and the integration by parts yields that

$$\int_0^{\beta} t\,P(t)\,\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt = P(\beta)\,\frac{\sin n\beta}{n} - P(0+)\int_0^{\beta}\frac{\sin nu}{nu}\,du$$

$$-\int_0^{\beta} C_n(t)\,dP(t),$$

where $C_n(t) = \dfrac{\sin nt}{n} + \displaystyle\int_t^{\beta}\frac{\sin nu}{nu}\,du.$

Therefore $$A_n(x) = \frac{2}{\pi}\int_{\beta}^{\pi}\phi(t)\cos nt\,dt + \frac{2}{\pi}\,\phi(\beta)\,\frac{\sin n\beta}{n}$$

$$-\frac{2}{\pi}\int_0^{\beta} C_n(t)\,dp(t) - \frac{2}{\pi}\,P(0+)\int_0^{\beta}\frac{\sin nu}{nu}\,nu$$

$$=\sum_{i=1}^{4} Q_n^{(i)}, \text{ say.} \qquad \ldots(4.1)$$

Now, in view of Lemma 4, we consider the summability $|E, q|$ $(q>0)$ of $\Sigma A_{n+1}(x)$. In this connection, we first observe that the proof of $\sum Q_{n+1}^{(i)} \in |E, q|$ $(q>0)$ for $i = 1$ and 2 is contained in Lemmas 1 and 2 respectively and

$$\sum Q_{n+1}^{(3)} \in |E, q|\ (q>0)$$

whenever (2.4 b) holds if, uniformly in $0 < t < \beta$,

$$\sum (1+q)^{-n-1}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\,C_{m+1}(t)\right| = O\{\log k/t)\}. \qquad \ldots(4.2)$$

Splitting up the above summation into $\sum_{n=0}^{T}$ and $\sum_{T+1}^{\infty}$, we obtain that

$$\sum_{n=0}^{T} = O\{\log (k/t)\}, \text{ uniformly in } 0 < t < \beta,$$

by using $C_n(t) = O(n^{-1})$.

And, by using $C_n(t) = \frac{\sin nt}{n} + O\{n^{-2} t^{-1}\}$

$$\text{we obtain that} \sum_{n>T} = \sum_{T+1}^{\infty} (1+q)^{-n-1} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin (m+1) t}{m+1} \right|$$

$$+ O(t^{-1}) \sum_{T+1}^{\infty} (1+q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (m+1)^{-2}$$

$$= O\{\log (k/t)\} + O(t^{-1}) \sum_{T+1}^{\infty} n^{-2} \qquad \text{(by Lemma 2)}$$

$$= O\{\log) (k/t)\}$$

uniformly in $0 < t < \beta$. Collecting the results, the proof of (4.2) may be completed.

We, now, observe that the sufficiency of (2.4 a) is trivial and the necessity is included in Theorem 3, which shall be proved after Theorem 2.

Proof of Theorem 2—Proceeding as in Theorem 1 with $\beta = \pi$, we obtain that

$$A_n(x) = -\frac{2}{\pi} \int_0^{\pi} C_n(t)\, dp(t) - \frac{2}{\pi} P(0+) \int_0^{\pi} \frac{\sin nu}{nu} du$$

$$= M_n + N_n, \text{ say.} \qquad ...(4.3)$$

$$\text{However, if } \sum_{n=0}^{\infty} (1+q)^{-n-1} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} y_{m+1} C_{m+1}(t) \right| = O\{\log \log (k/t)\}, \qquad ...(4.4)$$

uniformly in $0 < t < \pi$, then $\sum_{n=0}^{\infty} M_{n+1} y_{n+1} \in |E,q|$, whenever (2.10) holds.

By using $C_n(t) = O(n^{-1})$, it may be observed that

$$\sum_{n=0}^{T} = O\{\log \log (k/t)\}, \qquad ...(4.5)$$

uniformly in $0 < t < \pi$.

And, by using $C_n(t) = \frac{\sin nt}{n} + O\left\{t^{-1} n^{-2}\right\}$

we obtain that $\sum_{n>T} = \sum_{T+1}^{\infty} (1+q)^{-n-1} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} y_{m+1} \frac{\sin (m+1)t}{m+1} \right|$

$$+O(t^{-1}) \sum_{T+1}^{\infty} (1+q)^{-n-1} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} y_{m+1} (m+1)^{-2}$$

$$= \sum_{T+1}^{\infty} \frac{(1+q)^{-n-1}}{n+1} \left| \sum_{m=0}^{n} \binom{n+1}{m+1} q^{n-m} y_{m+1} \sin (m+1)\, t \right| + O(t^{-1})$$

$$\sum_{T+1}^{\infty} n^{-2} = O\left\{ t^{-1} \left(\log \frac{2\pi}{t}\right)^{-1} \right\} \sum_{T+1}^{\infty} n^{-3/2} + O(1) \qquad \text{(by Lemma 3)} \qquad \ldots(4.6)$$

$$= O(1), \text{ uniformly in } 0 < t < \pi.$$

Thus combining (4.5) and (4.6), the proof of (4.4) may be completed. We, now, prove that condition (2.4a) is necessary and sufficient. We obtain from (4.1) and (4.2) that (2.9) holds if and only if

$$(**) \qquad \sum_{n=0}^{\infty} N_{n+1}\, y_{n+1} \in |\, E, q \,| \;\; (q > 0).$$

And we observe that the sufficiency of (2.4 a) for (**) is trivial while the necessity of (2.4 a) for (**) is contained in Theorem 3, which shall be proved shortly.

Now we first show that (y_n) cannot be replaced by (y_n^{δ}) $(0 < \delta < 1)$. Let us take $f \in L[-\pi, \pi]$ with $2\pi-$ periodic for which

$$P(t) = (\log \log (k/t))^{-\alpha} \;\; (\alpha > 1) \text{ in } (0, \pi).$$

This is possible due to Lemma 5. Then (2.4a) and (2.10) hold and

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} (\log \log (k/t))^{-\alpha} \left(\cos nt - \frac{\sin nt}{nt}\right) dt = -\frac{2}{n\pi} (r_n + u_n)$$

where

$$r_n = \alpha \int_0^{\pi} \frac{\sin nt}{t \log \frac{k}{t} \left(\log \log \frac{k}{t}\right)^{1+\alpha}} dt = O\left\{ (\log n)^{-1} (\log \log n)^{-1-\alpha} \right\}$$

and, by Lemma 4 of the present author (1980),

$$u_n \sim \frac{\pi}{2} (\log \log n)^{-\alpha}.$$

Hence the proof follows.

The above example of f with $\alpha = 1$ proves that the condition (2.10) cannot be replaced by (2.11).

Proof of Theorem 3—We shall first prove that the series

$$\sum_{n=1}^{\infty} A_n(x)$$

diverges to $-\infty$.

Let δ be chosen with $0 < \delta \leqslant \pi$ such that $P(t) > 0$ for $0 < t < \delta$. We note that, for $t > 0$, $\phi_1(t)$ is an indefinite integral of

$$\frac{\phi(t)}{t} - \frac{1}{t^2}\int_0^t \phi(u)\,du = \frac{1}{t}\,(\phi(t) - \phi_1(t)) = \frac{P(t)}{t}. \qquad \ldots(4.7)$$

Hence, for $0 < t < \delta$, we have $\phi_1(\delta) - \phi_1(t) = \int_t^{\delta} \{P(u)/u\}\, du,$

$$\text{so that } \phi(t) = \phi_1(\delta) + P(t) - \int_t^{\delta} (P(u)/u)\,du. \qquad \ldots(4.8)$$

Not let the sequence of partial sums of the Fourier series be denoted by $\{s_n\} = \{s_n(x)\}$. Then

$$s_n = \frac{2}{\pi}\int_0^{\delta} \phi(t)\,\frac{\sin(n+\frac{1}{2})t}{t}\,dt + o(1) = \frac{2}{\pi}\int_0^{\delta}\{\phi_1(\delta)$$

$$+P(t)\}\,\frac{\sin(n+\frac{1}{2})t}{t}\,dt - \frac{2}{\pi}\int_0^{\delta}\frac{\sin(n+\frac{1}{2})t}{t}\,dt\int_t^{\delta}(P(u)/u)\,du + o(1)$$

$= I_1 - I_2 + o(1)$, say.

Since $\phi_1(\delta) + P(t)$ tends to a limit as $t \to 0+$, we have

$$I_1 = o(\log n).$$

$$\text{Now } I_2 = \frac{2}{\pi}\int_0^{\delta}\frac{P(u)}{u}\,du\int_0^{u}\frac{\sin(n+\frac{1}{2})t}{t}\,dt$$

$$= \frac{2}{\pi}\int_0^{\delta}(P(u)/u)\,X((n+\tfrac{1}{2})u)\,du, \qquad \ldots(4.9)$$

where $X(v) = \int_0^{v}((\sin w)/w)\,dw.$

Now, there is a (strictly) positive constant, say h, such that

$X(u) > hu \quad (0 \leqslant u \leqslant 1);$

$X(u) \geqslant h \quad (u > 1).$

Since $P(u) > 0$ for $0 < u < \delta$, we deduce from (4.9) that

$$I_2 \geqslant \frac{2h\,(n+\frac{1}{2})}{\pi} \int_0^{1/(n+\frac{1}{2})} P(u)\,du + \frac{2h}{\pi} \int_{1/(n+\frac{1}{2})}^{\delta} (P(u)/u)\,du$$

$$\sim \frac{2h}{\pi} P(0+) \log n,$$

as $n \to \infty$. The assertion is now evident.

Now to prove that $\sum_{n=1}^{\infty} y_n A_n(x)$ diverges to $-\infty$, we again suppose only that $P(t)$ tends to a positive limit as $t \to 0+$. The above argument shows that there is a strictly positive constant, say c, such that for all sufficiently large n, say for $n \geqslant n_0$, we have $\quad s_n \leqslant -c \log n.$ ...(4.10)

Now, for $N > n_0$,

$$\sum_{n=1}^{N} y_n A_n(x) = \sum_{n=1}^{N} y_n (s_n - s_{n-1}) = \left\{ -y_1 s_0 + \sum_{n=1}^{n_0-1} s_n (y_n - y_{n+1}) \right\}$$

$$+ \left\{ \sum_{n=n_0}^{N-1} s_n (y_n - y_{n+1}) + s_N y_N \right\}.$$

The first curly bracket is a constant. Since $y_n - y_{n+1} > 0$, it follows from (4.10) that the second curly bracket

$$\leqslant -c \left\{ \sum_{n=n_0}^{N-1} \log n\,(y_n - y_{n+1}) + (\log N)\,y_N \right\}$$

$$\sim -c \log \log N,$$

as $N \to \infty$. Hence the result.

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