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ANNULAR PUNCH PROBLEM FOR AN ELASTIC LAYER OVERLYING AN ELASTIC FOUNDATION*

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An axisymmetric indentation problem for an elastic layer overlying an elastic foundation by an annular rigid punch is considered. This is a three-part-mixed boundary value problem and is solved by the method of Shibuya *et al.* (1974, 1975). The quantities of physical interest are expressed in closed form in terms of the unknown coefficients a_n which are determined from the infinite set of simultaneous equations. These are solved numerically and first ten roots are considered. The variation of total load under the punch is shown graphically.

1. INTRODUCTION

One of the simplest three-part-mixed boundary value problem in the theory of elasticity is the contact problem for a flat annular rigid punch. Williams (1963), Cooke (1963), Noble (1963), Collins (1963) and Jain and Kanwal (1971) have shown that such mixed boundary value problems can be reduced to the solution of a Fredholm integral equation. The Fredholm equation is either solved by iterative techniques or by numerical techniques.

Shibuya *et al.* (1974, 1975) have proposed a noble method for solving indentation problems for infinite elastic medium by a flat rigid annular punch and thick elastic slab by a pair of flat annular punches. In this method, a simple fact that the normal pressure in the contact region is continuous at all points except the inner and outer edges of the punch is utilized to reduce the said problems to infinite set of simultaneous equations which is solved numerically.

The method of Shibuya *et al.* is extended to solve the title problem. The mixed boundary value problem is reduced to the solution of an infinite set of simultaneous equations. This set is solved numerically in section 5. The expressions for quantities of physical interest like total load under the punch and normal stress are derived in section 4. The variations of total load p^* with r_i/r_0 and for various values of $m = \mu_1/\mu_2$ and h are plotted in section 5.

The problem under discussion has application in soil mechanics, e.g., the annular punch may be regarded as a hollow pillar or as a chimney raised on a layered soil. Some conclusions are reported in section 5.

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2. Formulation of the Problem

We consider an infinite isotropic layer bounded by the planes $z=0$ and $z=-h$ of a cylindrical coordinate system (r,θ,z). The z-axis is directed downward and the semi-infinite isotropic space $z \geqslant 0$ is an elastic foundation which is in perfect bond with the layer (Fig. 1). The elastic properties of the layer and the foundation are assumed to be different. The free surface of the layer is indented with a rigid flat annular punch. In view of axial symmetry, the non-vanishing displacement components may be expressed in terms of the Boussinesq's stress functions $F(r,z)$ and $G(r,z)$ as

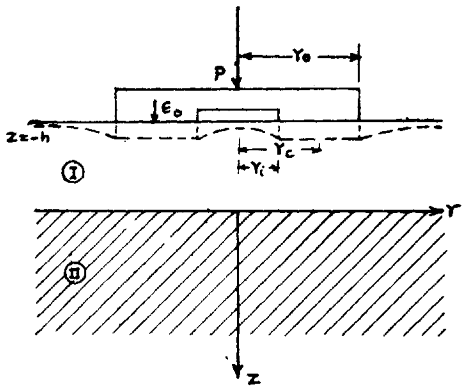

Fig. 1. Geometry of the Problem.

$$2\mu U_r = \frac{\partial F}{\partial r} + z\frac{\partial G}{\partial r} \quad \text{...(2.1)}$$

$$2\mu U_z = \frac{\partial F}{\partial z} + z\frac{\partial G}{\partial z} - (3-4\nu)G \quad \text{...(2.2)}$$

where μ and ν denote the shear modulus and the Poisson's ratio respectively. The stress components may be expressed as

$$\sigma_{zz} = \frac{\partial^2 F}{\partial z^2} + z\frac{\partial^2 G}{\partial z^2} - 2(1-\nu)\frac{\partial G}{\partial z} \quad \text{...(2.3)}$$

$$\sigma_{rz} = \frac{\partial^2 F}{\partial r \partial z} + z\frac{\partial^2 G}{\partial r \partial z} - (1-2\nu)\frac{\partial G}{\partial r}\,. \quad \text{...(2.4)}$$

Region I $(-h \leqslant z \leqslant 0)$

The two stress functions for this region may be expressed as

$$\left.\begin{aligned} F^{(1)}(r,z) &= \int_0^\infty [A_1(s)\,e^{-sz} + A_2(s)\,e^{sz}]\,J_0(sr)\,ds \\ G^{(1)}(r,z) &= \int_0^\infty [B_1(s)\,e^{-sz} + B_2(s)\,e^{sz}]\,J_0(sr)\,ds. \end{aligned}\right\} \quad \text{...(2.5)}$$

For this region, the quantities μ and ν are denoted by μ_1 and ν_1 and the displacement and stress components are $U_r^{(1)}$, $U_z^{(1)}$ and $\sigma_{zz}^{(1)}$ and $\sigma_{rz}^{(1)}$ respectively.

Region II ($z \geqslant 0$)

The quantities μ and ν for this region are denoted by μ_2 and ν_2 respectively and the two stress functions are

$$\left.\begin{aligned} F^{(2)}(r, z) &= \int_0^\infty C(s)\, e^{-sz} J_0(sr)\, ds \\ G^{(2)}(r, z) &= \int_0^\infty D(s)\, e^{-sz} J_0(sr)\, ds. \end{aligned}\right\} \qquad \ldots(2.6)$$

The displacement and stress components are $U_r^{(2)}, U_z^{(2)}$ and $\sigma_{zz}^{(2)}$, $\sigma_{rz}^{(2)}$ respectively.

3. Reduction to Infinite Simultaneous Equations

When the free surface $z=-h$ is indented by a rigid flat annular punch, the boundary conditions may be written as

$$U_z^{(1)}(r,-h)=\epsilon_0, \; r_i \leqslant r \leqslant r_0 \qquad \ldots(3.1)$$

$$\sigma_{zz}^{(1)}(r,-h)=0, \; 0 \leqslant r < r_i, \; r > r_0 \qquad \ldots(3.2)$$

$$\sigma_{rz}^{(1)}(r-h)=0, \; 0 \leqslant r < \infty \qquad \ldots(3.3)$$

Since the elastic layer and foundation are in perfect contact, the continuity conditions on $z=0$ must be satisfied. Thus on $z=0$, we have

$$\left.\begin{aligned} U_z^{(1)}(r,0) &= U_z^{(2)}(r,0), \; U_r^{(1)}(r,0) = U_r^{(2)}(r,0) \\ \sigma_{zz}^{(1)}(r,0) &= \sigma_{zz}^{(2)}(r,0), \; \sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0). \end{aligned}\right\} \qquad \ldots(3.4)$$

These continuity conditions will be satisfied if

$$\begin{aligned} 4(1-\nu_1)A_1 &= (m+3-4\nu_1)\, C - \{m(3-4\nu_2)(1-2\nu_1)-(1-2\nu_2)(3-4\nu_1)\}s^{-1}D \\ 4(1-\nu_1)A_2 &= (m-1)(3-4\nu_1)\, C + \{m(3-4\nu_2)(1-2\nu_1)-(1-2\nu_2)(3-4\nu_1)\}s^{-1}D \\ 4(1-\nu_1)B_1 &= \{m(3-4\nu_2)+1\}\, D \\ 4(1-\nu_1)B_2 &= 2(m-1)\, sC + (m-1)(3-4\nu_2)D \end{aligned} \qquad \ldots(3.5)$$

where $m = \mu_1/\mu_2$.

The boundary condition (3.3) is satisfied if

$$Cs = -[4\nu_1-3-m+(m-1)(1-2sh)e^{-2sh}]^{-1}\,[6(\nu_1+\nu_2)-4(1+2\nu_1\nu_2)+sh\{1+m(3-4\nu_2)\}+\{2(\nu_2-\nu_1)-sh(m-1)(3-4\nu_2)\}\,e^{-2sh}]D. \qquad \ldots(3.6)$$

The equations (3.5) and (3.6) express the unknown functions $A_1(s)$, $A_2(s)$...... $C(s)$ in terms of single unknown function $D(s)$. This unknown function is determined from the remaining boundary conditions (3.1) and (3.2). These conditions lead to the following triple integral equations:

$$\sigma_{zz}^{(1)}]_{z=-h} = \int_0^\infty sN(s)\, J_0(sr)\, ds = 0, \; 0 \leqslant r < r_i, \; r > r_0 \qquad \ldots(3.7)$$

$$U_z^{(1)}]_{z=-h} = \int_0^\infty [1+H(2sh)]N(s)\, J_0(sr)\, ds = -\frac{\mu_1\epsilon_0}{1-\nu_1}, \; r_i \leqslant r \leqslant r_0 \qquad \ldots(3.8)$$

where

$$\frac{D(s)}{4(1-\nu_1)} = \frac{k_3-\mu'(1-2sh)e^{-2sh}}{k_2k_3X(2sh)}\, e^{-sh}\, N(s) \qquad \ldots(3.9)$$

$$H(x)=-e^{-x}[p+q(1+x)^2+2t\,e^{-x}]\;[X(x)]^{-1} \quad \ldots(3.10)$$
$$X(x)=1+[p+q(1+x^2)]\;e^{-x}+te^{-2x} \quad \ldots(3.11)$$
$$p=k_1/k_2,\; q=-\mu'/k_3,\; t=-\mu' k_1/k_2\,k_3$$
$$k_1=(3-4\nu_1)-\mu(3-4\nu_2),\; k_2=1+m(3-4\nu_2)$$
$$k_3=m+3-4\nu_1,\; \mu'=m-1.$$

The present mixed boundary value problem is equivalent to:

(A) The circular stamp of radius r_0 is indented on the elastic layer and then the circular portion of radius $r_i(<r_0)$ is removed.

(B) The elastic layer is pressed by the infinite rigid plate with a circular hole of the radius r_i, and then the infinite portion beyond the ring of radius $r_0(>r_i)$ is released from pressing.

In the case (A), the singularity of $(\sigma_{zz}{}^{(1)})_{z=-h}$ at $r=r_0$ takes the form $(r_0^2-r^2)^{-1/2}$ before and after removing the circular portion. Similarly, in the case (B), the singularity of $(\sigma_{zz}{}^{(1)})_{z=-h}$ at $r=r_i$ takes the form $(r^2-r_i^2)^{-1/2}$. Therefore, the singularity in σ_{zz} for the problem under consideration will have the form $(r_0^2-r^2)^{-1/2}$ at $r=r_0$ and $(r^2-r_i^2)^{-1/2}$ at $r=r_i$. Thus in the region of annular stamp, $\sigma_{zz}{}^{(1)})_{z=-h}$ is assumed to have following form

$$\sigma^{(1)}_{zz\;z=-h}=\frac{-\epsilon_0 f(r)}{\sqrt{\{(r_0^2-r^2)(r^2-r_i^2)\}}}\;,\; r_i<r<r_0 \quad \ldots(3.12)$$

where $f(r)$ is an unknown function which is continuous in $r_i\leqslant r\leqslant r_0$ and non-zero at $r=r_0$ and $r=r_i$. It is convenient to define a new variable ϕ by the relation

$$\left.\begin{array}{l} 2r_c=r_i+r_0,\; 2b=r_0-r_i \\ 2\,r_c b\cos\phi=r_c^2+b^2-r^2. \end{array}\right\} \quad \ldots(3.13)$$

The variable $\phi=0$ and π at $r=r_i$ and r_0 respectively. The function $f(r)$ can now be expressed by the following Fourier series:

$$f(r)=\sum_{n=0}^{\infty} a_n'\cos n\phi,\; (r_i\leqslant r\leqslant r_0) \quad \ldots(3.14)$$

where a'_n are the unknown coefficients to be determined later.

Using (3.13) and (3.14), eqn. (3.12) may be written as

$$\sigma_{zz}{}^{(1)})_{z=-h}=\frac{-\epsilon_0}{2r_c b}\sum_{n=0}^{\infty} a'_n\frac{\cos n\phi}{\sin\phi}\;,\; r_i<r<r_0. \quad \ldots(3.15)$$

Since $\sigma_{zz}{}^{(1)})_{z=-h}=0$ in $0\leqslant r<r_i$ and $r_0<r$, the Hankel inversion of eqn. (3.15) gives us

$$N(s)=-\frac{\epsilon_0}{2}\sum_{n=0}^{\infty} a'_n\int_0^{\pi}\cos n\phi\,J_0\left(s\sqrt{r_c^2+b^2-2r_c b\cos\phi}\right)d\phi.$$

The use of formula

$$\frac{1}{\pi}\int_0^{\pi}\cos n\phi\,J_0\left(s\sqrt{r_c^2+b^2-2\,r_c b\cos\phi}\right)d\phi=J_n(sr_c)\,J_n(sb)\quad\text{gives us}$$

$$N(s) = -\frac{\pi\epsilon_0}{2}\sum_{n=0}^{\infty} a'_n J_n(sr_c)\, J_n(sb). \qquad ...(3.16)$$

Substituting eqn. (3.16) into (3.8), we get

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1+H(2sh)]\, J_0(sr)\, J_n(sr_c)\, J_n(sb)ds = 1, \quad r_i \leqslant r \leqslant r_0$$

Recalling the formula (Erdélyi 1954a, pp. 101)

$$J_0(sr) = J_0(sr_c)\, J_0(sb) + 2\sum_{m=1}^{\infty} J_m(sr_c) J_m(sb) \cos m\phi$$

where

$$r = (r_c^2 + b^2 - 2r_c b \cos\phi)^{1/2}$$

the above equation may be written as

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1+H(2sh)]\, z_n(s)\, z_m(s)\, ds = \delta_{0,m} \; (m=0, 1,2,...) \qquad ...(3.17)$$

where

$$z_n(s) = J_n(r_0 s) J_n(sb), \text{ and}$$

$$a_n = \frac{(1-\nu_1)}{2\mu_1} a'_n \qquad ...(3.18)$$

and δ_0, m is the Kronecker's delta. Equation (3.17) represents a set of infinite linear simultaneous equations for determination of the coefficients a_n.

4. Quantities of Physical Interest

We can now express $\sigma_{zz}^{(1)}(r,-h)$ and $U_z^{(1)}(r,-h)$ in terms of the coefficients a_n. From (3.15) we get

$$\sigma_{zz}^{(1)}(r,-h) = -\frac{\mu_1\epsilon_0}{(1-\nu_1)br_c}\sum_{n=0}^{\infty} a_n \frac{\cos n\phi}{\sin\phi}, \quad r_i < r < r_0. \qquad ...(4.1)$$

The displacement $U_z^{(1)}$ in the region $r < r_i$ and $r > r_0$, is

$$U_z^{(1)}(r,-h) = \epsilon_0 \sum_{n=0}^{\infty} a_n \left[I_0^n + \int_0^{\infty} H(2sh) J_0(sr) Z_n(s)\, ds \right] \qquad ...(4.2)$$

$$\text{where } I_0^n = \int_0^{\infty} J_0(sr)\, Z_n(s)\, ds. \qquad ...(4.3)$$

It is interesting to note that the first term in the above equation coincides with that of elastic half space problem, while the second term can be evaluated numerically since $H(2sh)$ tends to zero for large values of s.

The integral I_0 may be evaluated using result (Erdélyi 1954b, pp. 53)

$$I_0^n = \begin{cases} \dfrac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\Gamma(\frac{1}{2})r_c}\left(\dfrac{b}{r_c}\right)^n F(\tfrac{1}{2},n+\tfrac{1}{2},\ n+1;\ \sin^2\phi)F(\tfrac{1}{2},n+\tfrac{1}{2},\ 1;\ \sin^2\psi), & 0\leqslant r<r_i \\ \dfrac{(-1)^n}{\pi r}\left\{\dfrac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right\}^2\left(\dfrac{br_c}{r^2}\right)^n F(n+\tfrac{1}{2},\ n+\tfrac{1}{2},\ n+1;\ \sin^2\phi)F(n+\tfrac{1}{2},\ n+\tfrac{1}{2},\ n+1:\ \sin^2\psi), & r_0<r \end{cases}$$

where $F(\alpha, \beta, r, x)$ is the Gauss hypergeometric series and

$$\begin{bmatrix}\psi\\ \phi\end{bmatrix} = \begin{cases} \frac{1}{2}\left[\sin^{-1}\left(\dfrac{r+b}{r_c}\right)\pm\sin^{-1}\left(\dfrac{r-b}{r_c}\right)\right], & 0\leqslant r<r_i \\ \frac{1}{2}\left[\sin^{-1}\left(\dfrac{r_0}{r}\right)\pm\sin^{-1}\left(\dfrac{r_i}{r}\right)\right], & r_0<r. \end{cases}$$

Moreover ϕ in the region $r_i\leqslant r\leqslant r_0$ is defined by

$$\phi = \cos^{-1}\left[\frac{r_c^2+b^2-r^2}{2r_cb}\right].$$

The total compressive load P on the punch is

$$P = -2\pi\int_{r_i}^{r_0} r\sigma_{zz}^{(1)}(r,-h)\,dr = \frac{2\pi\mu_1\epsilon_0 a_0}{1-\nu_1}. \qquad \ldots(4.5)$$

5. Numerical Results

To determine the unknown coefficients a_n, we must solve infinite set of simultaneous equations (3.17). A general element of this set may be represented by

$$A_{mn}=A_{nm}=\int_0^\infty[1+H(2sh)]\,Z_m(s)\,Z_n(s)\,ds. \qquad \ldots(5.1)$$

Using asymptotic formula for Bessel functions for large values of s, eqn. (5.1) may be rewritten as

$$A_{mn} = \int_0^\lambda [1+H(2sh)]\,Z_m(s)Z_n(s)\,ds + A'_{mn} \qquad \ldots(5.2)$$

where

$$A'_{mn}=\int_\lambda^\infty \frac{1}{\pi^2 b r_c s^2}\Big[\cos^2 sr_i+\{(-1)^n+(-1)^m\}\sin sr_0\cos sr_i + (-1)^{m+n+2}\sin^2 sr_0\Big]ds. \qquad \ldots(5.3)$$

Integrating by parts, we get

$$A'_{mn} = \frac{1}{\pi^2 b r_c}\Big[\lambda^{-1}\cos^2\lambda r_i + r_i\,si(2\lambda r_i) + \{(-1)^m+(-1)^n\}\{\lambda^{-1}\sin\lambda r_0\cos\lambda r_i + r_c\,ci(2\lambda r_c) - b\,ci(2\lambda b) + (-1)^{m+n+2}\{\lambda^{-1}\sin^2\lambda r_0 - r_0\,\mathrm{si}(2\lambda r_0)\}\Big]. \qquad \ldots(5.4)$$

where

$$si(x) = \int_{\infty}^{x} \frac{\sin t}{t}\, dt,\; ci\,(x) = \int_{\infty}^{x} \frac{\cos t}{t}\, dt.$$

The first integral of (5.2) is evaluated numerically using 16 point Gauss-Legendre formula. The upper limit λ is fixed equal to 100. The second term of (5.2) can be evaluated numerically using eqn. (5.4). Thus, the coefficient matrix $A\,(m,n)$ is known.

The outer radius r_0 of the annular punch is fixed equal to 1 0 and all the distances are now measured in terms of r_0, the unit of length. The inner radius r_i is made to vary from 0.1 to 0.9 in step of 0.2. The thickness h of the elastic layer is made to vary from 0.25 to 5.0 The ratio $m = \mu_1/\mu_2$ is made to vary from 0 to 2.0 while ν_1 and ν_2 are fixed equal to 0.33 and 0.25 respectively. A set of 15 equations in 15 unknowns is solved and it has been observed that coefficients a_n decrease rapidly for $n \geqslant 10$. Thus, only first ten roots are taken into consideration for the infinite set of simultaneous equations.

The variation of total load $p^* = \dfrac{(1-\nu_1)P}{4\,\mu_1\,\epsilon_0}$ with r_i/r_0 and (for $h = 1.0$) is plotted in Fig. 2. It is seen that the total load p^* does not change appreciably when the ratio r_i/r_0 is smaller than 0.6, while it changes rapidly for r_i/r_0 between 0.6 and 0.9. It is further observed that the value of total load p^* below punch decrease rapidly for a slight deviation of m from its ideal value $m = 0$ (for rigid foundation). Thus, it is important to take into consideration the elastic nature of even a very stiff foundation, particularly when elastic layer is not very thick. When $r_i \to 0$, the value of total load p^* coincides with the value of p^* for the corresponding problem of solid cylindrical punch, solved by Dhaliwal (1970).

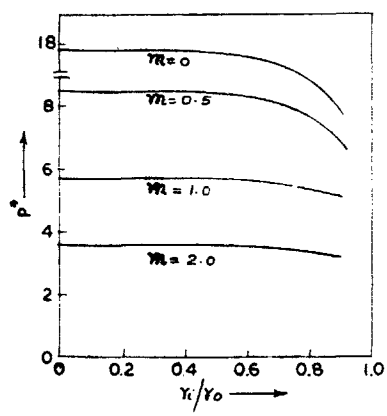

FIG 2. Variation of P^* with r_i/r_0 and m.

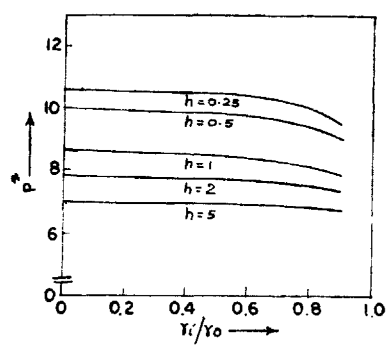

FIG. 3. Variation of P^* with r_i/r_0 and h.

The variation of total p^* with r_i/r_0 and h (for $m = 0.5$) is shown in Fig. 3. It is again observed that the value of p^* does not change appreciably when the ratio r_i/r_0

is smaller than 0.6, while it changes rapidly for r_i/r_0 between 0.6 and 0.9. Again, when $r_i \to 0$, the value of p^* coincides with the value of p^* for the corresponding problem of cylindrical punch, solved by Dhaliwal (1970).

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