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ON THE TYPE OF ANALYTIC DIRICHLET SERIES OF FAST GROWTH

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A coefficient characterization for the type and lower type of analytic functions of fast growth represented by Dirichlet series has been obtained. A decomposition theorem involving H_p-type and lower H_p-type ($p > 2$) is also established.

§1. Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \qquad \ldots(1)$$

where $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$, $s = \sigma + it$ (σ, t real variables) subject to

$$\limsup_{n\to\infty} n \mid \lambda_n = D < \infty \qquad \ldots(2)$$

represent a holomorphic function $f(s)$ in the half-plane $\sigma < A$ ($-\infty < A < \infty$) where

$$A = -\limsup_{n\to\infty} (\log \mid a_n \mid)/\lambda_n.$$

Again, if for $\sigma < A$

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b}} \mid f(\sigma + it) \mid, \quad m(\sigma) = \max_{n\geqslant 1} \{ \mid a_n \mid \exp(\sigma\lambda_n)\}$$

and

$$\lambda_{N(\sigma)} = \max \{\lambda_n \mid m(\sigma) = \mid a_n \mid \exp(\sigma\lambda_n)\}$$

where N and hence λ_N is a function of σ. It is well-known that $\log M(\sigma)$ is an increasing convex function of σ in $\sigma < A$. The result is analogous to the corresponding one (Titchmarsh 1950, 5.42, p. 174) for the analytic function represented by Taylor series.

§2. As might be expected, there are features peculiar to analytic functions represented by Dirichlet Series, not shared by analytic functions represented by power series. Much work has been investigated in the case of the latter series (Kapoor 1976, 1980; Linden 1970; Sons 1968 etc.). It is, moreover, concerned with an analytic function represented by Dirichlet series having infinite order in a sense defined in Nandan (1973; 1976, 1978, 1980a) but of finite order in an extension of that sense.

Further, let ρ be order (Nandan 1973) of $f(s)$, i.e.,

$$\lim_{\sigma\to A} \sup \log \log M(\sigma)/[-\log\{1-\exp(\sigma-A)\}] = \rho. \qquad \ldots(3)$$

If $f(s)$ is of fast growth, the function $f(s)$ is said to be of H_p-order $\rho(p)$ and of lower H_p-order $\lambda(p)$ if and only if (Nandan 1980b)

$$\lim_{\sigma\to A} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_p M(\sigma)}{-\log\{1-\exp(\sigma-A)\}} = \begin{matrix}\rho(p)\\ \lambda(p)\end{matrix} \qquad \ldots(4)$$

where we use the familiar abbreviation $l_p(x) = \log\log \ldots$ (p times) $x(p = 1, 2, 3, \ldots)$ observing that $l_p x > 0$ for real x after 2 stage. $\rho(p)$ is finite for some least positive integer $p > 2$ while it is infinite for smaller p. Thus $\rho(p)$ naturally extends the definition of ρ given in (3) which corresponds to $p = 2$. $\rho(p)$ is strictly positive as well as finite.

§3. If two of the functions have the same nonzero order, by confining to the notion of order only, it is not possible to compare their growth precisely. For this purpose type and lower type are defined as below:

If the function $f(s)$, defined by (1), is analytic for $\sigma < A$, having H_p-order $\rho(p)$ ($\rho(p) > 0$), it is said to be of H_p-type $T(p)$ and of lower H_p-type $\tau(p)$ if and only if.

$$\lim_{\sigma\to A} \begin{matrix}\sup\\ \inf\end{matrix} (l_{p-1}M(\sigma)/\{1-\exp(\sigma-A)\}^{-\rho(p)} = \begin{matrix}T(p)\\ \tau(p)\end{matrix} \quad (0 \leqslant \tau \leqslant T \leqslant \infty).$$

The function $f(s)$ is said to have growth $\{\rho, T\}$ in $\sigma < A$ if it is of H_p-order not exceeding $\rho(p)$ and type not exceeding $T(p)$ if of order $\rho(p)$. The function $f(s)$, analytic and of regular growth in $\sigma < A$ is said to be of perfectly regular growth if

$$T(p) = \tau(p).$$

In the present paper, we obtain a coefficient characterization for the type and lower type of the function of fast growth. We also give a decomposition theorem involving H_p-type and lower H_p-type. Throughout our discussion we take $p > 2$.

Theorem 1 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$ with H_p-order $\rho(\rho > 0)$, H_p-type T and lower H_p-type τ. If $f(s)$ satisfies,

$$\lim_{n\to\infty} (\lambda_n - \lambda_{n-1}) = \beta > 0 \qquad \ldots(5)$$

then

$$\lim_{\sigma \to A} \sup_{\inf} (l_{p-1} m(\sigma))/\{1 - \exp(\sigma - A)\}^{-\rho} = \frac{T}{\tau}. \qquad \text{...(6)}$$

The proof of the theorem follows in a straight forward manner by using the following relations Nandan (1980b, 1978, 1973) and the fact that $m(\sigma) \leqslant M(\sigma)$:

$$\log m(\sigma) = \log m(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(u)} du, \ -\infty < \sigma_1 < \sigma < A \qquad \text{...(7)}$$

and for every $\gamma < \beta$

$$M(\sigma) < m(\sigma)\left[1 + \frac{\gamma+1}{\gamma} N\left\{\sigma + \frac{1-\exp(\sigma-A)}{N(\sigma)}\right\}\right]\{1-\exp(\sigma-A)\}^{-1}. \qquad \text{...(8)}$$

For the function $f(s)$ defined by (1) and analytic in $\sigma < A$, set, for $\rho > 0$

$$V = \lim_{n \to \infty} \sup [\log^+\{ | a_n | \exp (A\lambda_n)\}]^\rho l_{p-2}\lambda_n/\lambda_n^\rho \qquad \text{...(9)}$$

where $\log^+(x) = \max(0, \log x)$, then we have the theorem.

Theorem 2 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$. If $0 < V < \infty$, the function $f(s)$ is of order ρ and type T if and only if $V = T$. If $V = 0$ or ∞, $f(s)$ is respectively of growth $\{\rho, 0\}$ or of growth not less than $\{\rho, \infty\}$, and conversely. V is the same as given in Sons (1968).

PROOF : Suppose first $V < \infty$. For given $\epsilon > 0$ and for all $n > n_0(\epsilon)$ we have, from (9)

$$\log | a_n | < B\, \lambda_n (l_{q-2}\lambda_n)^{-1/\rho} - A\lambda_n, \quad B = (V + \epsilon)^{1/\rho}, \qquad \text{...(10)}$$

or

$$l_{q-1}\lambda_n/[\log \lambda_n - \log^+\{A\lambda_n + \log | a_n | \}] < (1 + O(1))\,\rho.$$

Hence, utilizing the result (Nandan 1980b; Theorem 3) i.e.,

$$\rho(p) = \lim_{n \to \infty} \sup (l_{p-1}\lambda_n)/\{\log \lambda_n - \log^+(A\lambda_n + \log | a_n |)\}$$

$f(s)$ is of order at most ρ. Similarly if $V > 0$, the order of $f(s)$ is atleast ρ. Thus, if $0 < V < \infty$, $f(s)$ is of order ρ.

Let $0 \leqslant T < \infty$, from the definition of T it follows that for given $\epsilon > 0$ and for all σ such that $A > \sigma > \sigma_0 = \sigma_0(\epsilon)$, $T + \epsilon = \overline{T}$

$$\log M(\sigma) < e_{p-2}[\bar{T}\{1 - \exp(\sigma - A)\}^{-\rho}]$$

where $e_p x = \exp \exp \ldots (p \text{ times})\, x$.

Using Cauchy's estimate, we have for all n and for all σ such that $A > \sigma > \sigma_0 = \sigma_0(\epsilon)$,

$$\log^+ \{ \mid a_n \mid \exp(A\lambda_n)\} < e_{p-2}[\bar{T}\{1 - \exp(\sigma - A)\}^{-\rho}] - (\sigma - A)\,\lambda_n. \quad \ldots(11)$$

Choose,

$$e_{p-2}\,[\bar{T}\{1 - \exp(\sigma - A)\}^{-\rho}] = [\{l_{p-2}(\lambda_n/\rho)\}/\bar{T}]^{-(1+\rho)/\rho}\lambda_n/\rho.$$

Then, for all $n > n_0 = n_0(\sigma_0)$, the inequality (11) is reduced to

$$\log^+\{ \mid a_n \mid \exp(A\lambda_n)\} < \{(1 + O(1))^{-1/\rho} + O(1)\}\, \lambda_n \bar{T}^{1/\rho}(l_{p-2}\lambda_n)^{-1/\rho}$$

or

$$[(\log^+\{ \mid a_n \mid \exp(A\lambda_n)\})/\lambda_n]^{\rho}\, l_{p-2}\lambda_n < \bar{T}\{(1 + o(1))^{-1/\rho} + o(1)\}^{\rho}.$$

Passing to limits

$$V \leqslant T. \quad \ldots(12)$$

Next suppose that $0 \leqslant V < \infty$,

$$M(\sigma) \leqslant P(n_0) + \Big(\sum_{n=n_0+1}^{R} + \sum_{n=1+R}^{\infty}\Big) \mid a_n \mid \exp(\sigma\lambda_n)$$

where $P(n_0)$ stands for the sum of first n_0 terms.
Making use of the inequality (10) we have

$$M(\sigma) < O(1) + \Big(\sum_{n=1+n_0}^{R} + \sum_{n=1+R}^{\infty\dagger}\Big) \exp[B\lambda_n(l_{p-2}\lambda_n)^{-1/\rho}$$
$$+ (\sigma - A)\,\lambda_n]. \quad \ldots(13)$$

It can be easily found out that

$$\max_{0<\lambda_n<\infty} B\lambda_n(l_{p-2}\lambda_n)^{-1/\rho} + (\sigma - A)\,\lambda_n$$
$$= (B/\rho)\{(A - \sigma)/B\}^{1+\rho} e_{p-2}\{(A - \sigma)/B\}^{-\rho}.$$

Comparing the series $f(s)$ with the convergent series $\Sigma \exp\{(\sigma - A)\,\lambda_n/2\}$ and using eqn. (2), the last part† on the right-hand side of the inequality (13) can be made equal to a bounded quantity as $\sigma \to A$ for all $n > R = [(D + \epsilon)\, e_{p-2}\{(A - \sigma)/2B\}^{-\rho}]$.

Thus in view of the above approximation, the inequality (13) is reduced to

$$M(\sigma) < R \exp\left[\{(A-\sigma)/B\}^{1+\rho}\, \frac{B}{\rho}\, e_{p-2}\left(\frac{B}{A - \sigma}\right)^{\rho}\right] + O(1)$$

or

$$l_2 M(\sigma) < O(1) + e_{p-3}(B/(A - \sigma))^{\rho} + (1 + \rho) \log (A - \sigma)$$

or

$$\frac{l_{p-1} M(\sigma)}{\{1 - \exp (\sigma - A)\}^{-\rho}} < B^{\rho} [(A - \sigma)/\{1 - \exp (\sigma - A)\}]^{-\rho} + o(1).$$

Passing to limits, we get

$$T \leqslant V. \qquad \ldots(14)$$

Equations (12), (14) and the fact that for $0 < V < \infty$, $f(s)$ is of order ρ, prove the first part of the theorem. Now, if $V = 0$, then $f(s)$ is of order atmost ρ and (14) gives that if $f(s)$ is of order ρ then its type is zero (since $V = 0$). Hence, if $V = 0$, then $f(s)$ is of growth $\{\rho, 0\}$. Similarly, if $V = \infty$, then $f(s)$ is of order atleast ρ and (12) shows that if $f(s)$ is of order ρ then $T = \infty$. Hence, if $V = \infty$, $f(s)$ is of growth not less than $\{\rho, \infty\}$. The converse also follows in a similar manner.

Remark : For $p = 2$, the result is $V = (1 + \rho)^{1+\rho} T/\rho^{\rho}$

where

$$V = \lim_{n \to \infty} \sup [\log^{+} \{ \mid a_n \mid \exp (A\lambda_n)\}]^{1+\rho}/\lambda_n^{\rho}, \quad (0 < \rho < \infty)$$

which is due to Nandan (1978).

We observe that the result analogous to that of the previous theorem does not always hold for the lower H_p-type τ. For, consider

$$f(s) = \sum_{n=1}^{\infty} \exp \{n \exp (l_{p-1}\, n) + ns\} + \Sigma \exp \{n \exp (-l_{p-1}\, n) + 2ns\}$$

$$= f_1(s) + f_2(s), \text{ (say)}. \qquad \ldots(15)$$

Here $A = 0$, $\log m(\sigma, f_2) \sim -2\sigma l_{p-1} \{-\log (-2\sigma)\}$ as $\sigma \to 0$.

Since $\log M(\sigma, f) \sim \log M(\sigma, f_2)$, we have that $\rho = \lambda = 1$ and $\tau = \frac{1}{2}$ (using Theorem 1). But if

$$C(n) = (l_{p-2}\lambda_n \log^{+} \mid a_n \mid)/\lambda_n, \text{ then}$$

$$\lim_{n \to \infty} C(2n) = 1, \quad \lim_{n \to \infty} C(2n + 1) = 0$$

which shows that

$$\lim_{n \to \infty} \inf B(n) = 0 \neq \tau.$$

However the following relation always holds.

Theorem 3 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$ having H_p-order $\rho(0 < \rho)$ and lower H_p-type $\tau(0 \leqslant \tau \leqslant \infty)$. If $\{n_K\}_{K=1}^{\infty}$ is an increasing sequence of natural numbers, then

$$\tau \geqslant \lim_{k\to\infty} \inf \, [\log^+ \{ \mid a_{n_k} \mid \exp(A\lambda_{n_k})\}^{1/\lambda_{nk}}]^{\rho} \, l_{p-2} \, \lambda_{n_{k-1}}. \qquad \ldots(16)$$

Proof : Let

$$E = \lim_{k\to\infty} \inf \, [\log^+ \{ \mid a_{n_k} \mid \exp(A\lambda_{n_k})]\}^{1/\lambda_{nk}}]^{\rho} \, l_{p-2} \, \lambda_{n_{k-1}}$$

Then $0 \leqslant E \leqslant \infty$. First suppose that $0 < E < \infty$. Then for every ϵ such that $E > \epsilon > 0$,

$$\log \mid a_{n_k} \mid > \lambda_{n_k} \{(E - \epsilon)/l_{p-2} \lambda_{n_{k-1}}\}^{1/\rho} - A\lambda_{n_k}$$

for all $k > k_0(\epsilon)$.

Let $\sigma_k - A = -\frac{1}{e} \{(E - \epsilon)/l_{p-2}\lambda_{n_{k-1}}\}^{1/\rho}$, for $k = 1, 2, 3, \ldots$

If $k > k_0$ and $\sigma_k \leqslant \sigma \leqslant \sigma_{k+1}$, then

$$\log M(\sigma) \geqslant \log \mid a_{n_k} \mid + \sigma\lambda_{n_k}$$

$$> \left(1 - \frac{1}{e}\right)(A - \sigma)e_{p-2} \{(E - \epsilon)(A - \sigma)^{-\rho}\}$$

or $$l_2 M(\sigma) > (1 + o(1)) \, e_{p-3} \{(E - \epsilon)(A - \sigma)^{-\rho}\}$$

or $$l_{p-1} M(\sigma) > (E - \epsilon)(A - \sigma)^{-\rho} + o(1).$$

Dividing by $(1 - \exp(\sigma - A))^{-\rho}$ and passing to limits, we get

$$\tau \geqslant E$$

If $E = 0$, the result follows trivially and when $E = \infty$, the above arguments with an arbitrarily large number in place of $(E - \epsilon)$ given $\tau = \infty$.

Remark : For $p = 2$, the result in (16) was obtained by Nandan (1980a) as

$$\tau \frac{(1 + \rho)^{1+\rho}}{\rho^{\rho}} \geqslant \lim_{k\to\infty} \inf \, \lambda_{n_{k-1}} [\log^+ \{ \mid a_{n_k} \mid \exp(s\lambda_{n_k})\}^{1/\lambda_{nk}}]^{1+\rho}.$$

Theorem 4 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$ having H_p-order $\rho(0 < \rho)$, H_p-type T and lower H_p-type $\tau(0 \leqslant \tau \leqslant T \leqslant \infty)$, If $f(s)$ satisfies (5) and

$$\psi(n) \equiv (\log | a_n/a_{n+1} |)/(\lambda_{n+1} - \lambda_n) \qquad ...(17)$$

is a non-decreasing function of n for large n, then

$$\tau \leqslant \liminf_{n\to\infty} [\log^+ \{ | a_n | \exp (A\lambda_n)\}^{1/\lambda_n}]^\rho \, l_{p-2}\, \lambda_n. \qquad ...(18)$$

PROOF : Let $0 < \tau < \infty$, then from (6), for given ϵ such that $\tau > \epsilon > 0$, we have

$$\log m(\sigma) > e_{p-2} \{(\tau - \epsilon)(1 - \exp (\sigma - A))^{-\rho}\}$$

for all $\sigma > \sigma_0 = \sigma_0(\epsilon)$. Let $\sigma > \sigma_0$ $(\sigma < A)$ and let n_1 and $n_2 (n_2 - 1 \geqslant n_1)$ be two consecutive maximum terms, then

$$\log | a_{n_2} | + \sigma\lambda_{n_2} > e_{p-2} [(\tau - \epsilon) \{1 - \exp (\sigma - A)\}^{-\rho}]$$

for all σ satisfying $\psi(n_2 - 1) \leqslant \sigma < \psi(n_2)$. Let $n_1 \leqslant n < n_2 - 1$, it is obvious that

$$\psi(n_1) = \psi(n_1 + 1) = \ldots = \psi(n) = \ldots = \psi(n_2 - 1),$$

and that $\quad | a_n | \exp (\sigma_n) = | a_{n_2} | \exp (\sigma\lambda_{n_2})$ for $\sigma = \psi(n)$.

Hence,

$$[\log^+ \{ | a_n | \exp (A\lambda_n)\}^{1/\lambda_n}]^\rho \, l_{p-2}\, \lambda_n$$

$$> [\{A - \psi(n)\} \lambda_n + e_{p-2} (\tau-\epsilon) \{1 - \exp (\psi(n)-A)\}^{-\rho})]^\rho \, \frac{l_{p-2}\lambda_n}{\lambda_n^\rho}.$$

If σ satisfies that

$$e_{p-2}[(\tau - \epsilon) \{1 - \exp (\sigma - A)\}^{-\rho}] = \frac{\lambda_n}{\rho} \{(\tau - \epsilon)^{-1} l_{p-2}(\lambda_n/\rho)\}^{-(1+(1/\rho))},$$

then

$$[\log^+ \{ | a_n | \exp (A\,\lambda_n)\}^{1/\lambda_n})^\rho \, l_{p-2}\, \lambda_n > (\tau - \epsilon) [o (1) + (1 + o (1))^{-1/\rho}]^\rho.$$

Passing to limits, we get (18).

The inequality holds obviously if $\tau = 0$. If $\tau = \infty$, the above arguments can be carried with an arbitrarily large number in place of $(\tau - \epsilon)$ to give that the right-hand side of (18) is equal to infinity.

Theorem 5 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp (s\lambda_n)$ be analytic in $\sigma < A$ having H_p-order $\rho(0 < \rho)$ and lower H_p-type $\tau(0 \leqslant \tau \leqslant \infty)$. If it satisfies (5), (17) and that

$l_{p-2}\,\lambda_n \sim l_{p-2}\,\lambda_{n+1}$ as $n \to \infty$, then

$$\tau = \liminf_{n\to\infty} [\log^+ \{ | a_n | \exp (A\,\lambda_n)\}^{1/\lambda_n})^\rho \, l_{p-2}\, \lambda_n. \qquad ...(19)$$

Note : Theorems 3 and 4 give this theorem.

Our next theorem gives a coefficient characterization of the lower type τ which holds for a wider subclass of analytic functions.

Theorem 6 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$ with H_p-order $\rho(0 < \rho)$ and H_p-type $\tau(0 \leqslant \tau \leqslant \infty)$. If $f(s)$ satisfies (5) and $\{n_k\}_1^{\infty}$ is the range of $N(\sigma)$ such that $l_{p-2}\lambda_{n_{k-1}} \sim l_{p-2}\lambda_{n_k}$ as $k \to \infty$, then

$$\tau = \max_{\{n_k\}} [\lim_{k\to\infty} \inf [\log^+ \{ | a_{n_k} | \exp(A\lambda_{n_k})\}^{1/\lambda_{n_k}}]^{\rho} l_{p-2}\lambda_{n_{k-1}} \qquad ...(20)$$

where maximum is taken over all increasing sequences $\{n_k\}_1^{\infty}$ of natural numbers.

PROOF : Consider the function $g(s) = \Sigma a_{n_k} \exp(s\lambda_{n_k})$, $\{n_k\}_1^{\infty}$ being the range of $N(\sigma)$ for $f(s)$. It is easily seen that $g(s)$ and $f(s)$ have the same maximun term for every s in the half-plane $\sigma < A$; hence $g(s)$ is of H_p-order ρ and lower H_p-type τ. Since $\psi(n_k)$ are jump points of the rank of $f(s)$, they form an increasing sequence. Hence, using Theorem 5, we have

$$\tau = \lim_{k\to\infty} \inf [\log^+ \{ | a_{n_k} | \exp(A\lambda_{n_k})\}^{1/\lambda_{n_k}}]^{\rho} l_{p-2}\lambda_{n_{k-1}}. \qquad ...(21)$$

But, by Theorem 3

$$\tau \geqslant \max_{\{n_k\}} \lim\inf [\log^+ \{ | a_{n_k} | \exp(A\lambda_{n_k})\}^{1/\lambda_{n_k}}]^{\rho} l_{p-2}\lambda_{n_{k-1}}. \qquad ...(22)$$

Combining (21) and (22) gives the theorem.

Remark : For $p = 2$, the above theorem was obtained (Nandan 1980a) as

$$\frac{(1+\rho)^{1+\rho}}{\rho^{\rho}} \tau = \max_{\{n_k\}} [\lim_{k\to\infty} \inf \lambda_{n_{k-1}} \{\log^+(| a_{n_k} | \exp(A\lambda_{n_k}))^{1/\lambda_{n_k}}\}]^{1+\rho}$$

under the similar conditions viz, (5) and $\lambda_{n_{k-1}} \sim \lambda_{n_k}$ as $k \to \infty$, where miximum has the same meaning.

Theorem 7 — Let $f(z) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be analytic in $\sigma < A$ having H_p-order $\rho(0 < \rho)$, H_p-type T and lower H_p-type $\tau(0 < \tau < T < \infty)$. If μ be such that $\tau < \mu < T$, then

$$f(s) = g(s) + h(s)$$

where $g(s)$ is of growth $\{\rho, \mu\}$ and $h(s) = \sum_{k=1}^{\infty} a_{m_k} \exp(s\lambda_{m_k})$, $a_{m_k} \neq 0$ for all k satisfies

$$\tau \geqslant \mu \liminf_{k \to \infty} \left(\frac{l_{p-2} \lambda_{m_k}}{l_{p-2} \lambda_{m_{k+1}}} \right)^{(p+F)/(p+1)}$$

where $F = 0$ for $p = 2$, otherwise 1.

The proof is similar to that of Theorem 6 (Nandan 1973) and hence omitted.

Remark : The theorem generalizes the result obtained by Nandan (1980a) for $p = 2$.

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