Indian J. pure appl. Math., **13**(5): 581-585, May 1982

STRESSES PRODUCED IN AN INITIALLY STRESSED ELASTIC HALF-SPACE DUE TO A MOVING LOAD

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(*Received* 3 *April* 1981; *after revision* 31 *July* 1981)

The two dimensional problem of distribution of incremental stresses produced by a pulse of shearing force moving over the boundary of a transversely isotropic elastic half-space has been investigated in this paper. It has been shown that the incremental shearing stress attains the maximum value at a point directly below the point on the boundary where the moving shearing force acts.

1. INTRODUCTION

The two dimensional problems of distribution of stresses produced by a force moving on the boundary of an elastic half-space have been studied by Sneddon (1951), Mukherjee (1969), Sur (1963) and others.

Sneddon (1951) discussed the problem in which a pulse of pressure was moving uniformly along the boundary of an isotropic half-space. Mukherjee (1969) considered the problem in which a load was moving on the rough boundary of a transversely isotropic elastic half-space. Sur (1963) has solved the problem in which a shearing force is moving uniformly over the boundary of a semi-infinite transversely isotropic elastic solid.

But none of the authors has considered the pre-stressed condition of the medium. Here we have derived the expressions of incremental stresses produced by a shearing force moving over the boundary of a transversely isotropic elastic half-space.

2. FORMULATION OF THE PROBLEM

We consider a two-dimensional motion confined to the x-z plane where the x axis is taken in the direction in which shearing force moves and the z axis points into the semi-infinite medium. Let the transversely isotropic elastic half-space $z \geqslant 0$ be under uniform initial compressive stresses S_{11} and S_{33} acting along x and z directions respectively.

The stress equations of motion (Biot 1965) in the absence of body forces and under initial stress $P = S_{33} - S_{11}$ are

$$\left.\begin{aligned}&\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{13}}{\partial z}+P\frac{\partial \omega_y}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2}\\ \text{and}\quad&\\ &\frac{\partial s_{31}}{\partial x}+\frac{\partial s_{33}}{\partial z}+P\frac{\partial \omega_y}{\partial x}=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\quad\ldots(1)$$

where $(u, 0, w)$ are the displacement components, s_{ij} are the incremental stress components and $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$.

The incremental stress components in terms of displacements (Biot 1965) may be written as

$$\left.\begin{aligned}&s_{11}=(c_{11}+P)\frac{\partial u}{\partial x}+(c_{13}+P)\frac{\partial w}{\partial z}\\ &s_{33}=c_{33}\frac{\partial w}{\partial z}+c_{13}\frac{\partial u}{\partial x}\\ \text{and}\quad&\\ &s_{13}=s_{31}=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\end{aligned}\right\}\quad\ldots(2)$$

where c_{11}, c_{13},... etc. being elastic constants of the medium considered.

Using (2) the stress equations of motion (1) are

$$\left.\begin{aligned}&(c_{11}+P)\frac{\partial^2 u}{\partial x^2}+\left(c_{44}+\frac{P}{2}\right)\frac{\partial^2 u}{\partial z^2}+\left(c_{13}+c_{44}+\frac{P}{2}\right)\frac{\partial^2 w}{\partial x\,\partial z}=\rho\frac{\partial^2 u}{\partial t^2}\\ \text{and}\quad&\\ &\left(c_{44}-\frac{P}{2}\right)\frac{\partial^2 w}{\partial x^2}+c_{33}\frac{\partial^2 w}{\partial z^2}+\left(c_{13}+c_{44}+\frac{P}{2}\right)\frac{\partial^2 u}{\partial x\partial z}=\rho\frac{\partial^2 w}{\partial t^2}.\end{aligned}\right\}\quad\ldots(3)$$

The boundary conditions for the motion produced by a pulse of shearing force F moving uniformly with a velocity c along x-axis may be written as

$$\left.\begin{aligned}\Delta f_x &= -F\,\delta\,(x-ct)\\ \Delta f_z &= 0\end{aligned}\right\}\ \text{at } z=0\quad\ldots(4)$$

where Δf_x and Δf_z are the x and z components of the incremental boundary forces per unit initial area respectively.

In terms of displacements these boundary forces are (Biot 1965)

$$\left.\begin{aligned}\Delta f_x &= \left(c_{44}+\frac{P}{2}\right)\frac{\partial u}{\partial z}+\left(c_{44}+\frac{P}{2}-S_{33}\right)\frac{\partial w}{\partial x}\\ \Delta f_z &= c_{33}\frac{\partial w}{\partial z}+\left(c_{13}+S_{33}\right)\frac{\partial u}{\partial x}.\end{aligned}\right\}\quad\ldots(5)$$

Therefore the boundary conditions (4) may be rewritten as

$$\left.\begin{aligned}&\left(c_{44}+\frac{P}{2}\right)\frac{\partial u}{\partial z}+\left(c_{44}+\frac{P}{2}-S_{33}\right)\frac{\partial w}{\partial x}=-F\,\delta\,(x-ct)\\ &c_{33}\frac{\partial w}{\partial z}+\left(c_{13}+S_{33}\right)\frac{\partial u}{\partial x}=0\end{aligned}\right\}\ \text{at } z=0,\quad\ldots(6)$$

where $\delta\,(x)$ stands for Dirac delta function of the argument x.

3. Solution of the Problem

The solutions of the equations (3) may be assumed as

$$\left.\begin{aligned} u &= \int_0^\infty A e^{-kqz} \cos k\,(x-ct)\,dk \\ w &= \int_0^\infty B e^{-kqz} \sin k\,(x-ct)\,dk \end{aligned}\right\} \qquad \ldots(7)$$

where q is a parameter independent of k.

Substituting (7) into (3) we have

$$A\left[\rho c^2 - \left(c_{11} + P\right) + \left(c_{44} + \frac{P}{2}\right)q^2\right] + B\left(c_{13} + c_{44} + \frac{P}{2}\right) q = 0 \qquad \ldots(8)$$

and

$$A\left(c_{13} + c_{44} + \frac{P}{2}\right) q + B\left(c_{44} - \frac{P}{2} - \rho c^2 - c_{33}\, q^2\right) = 0. \qquad \ldots(9)$$

For non-zero values of A and B we have, from (8) and (9)

$$q^4 + \left[\frac{\rho c^2 - \left(c_{44} - \frac{P}{2}\right)}{c_{33}} + \frac{\rho c^2 - \left(c_{11} + P\right)}{c_{44} + P/2} + \frac{\left(c_{13} + c_{44} + \frac{P}{2}\right)^2}{c_{33}\left(c_{44} + P/2\right)}\right] q^2 + \frac{\left[\rho c^2 - (c_{11} + P)\right]\left[\rho c^2 - \left(c_{44} - \frac{P}{2}\right)\right]}{c_{33}\left(c_{44} + P/2\right)} = 0. \qquad \ldots(10)$$

Let q_1^2 and q_2^2 be the two roots of eqn. (10). q_1 and q_2 will be real if q_1^2 and q_2^2 are both positive.

So for real q_1 and q_2

$[\rho c^2 - (c_{11} + P)]\,[\rho c^2 - (c_{44} - \frac{1}{2} P)]$ should be positive. Hence c must be greater than or less than both $\sqrt{(c_{11} + P)/\rho}$ and $\sqrt{(c_{44} - \frac{1}{2} P)/\rho}$. But if c is greater than both $\sqrt{(c_{11} + P)/\rho}$ and $\sqrt{(c_{44} - \frac{1}{2} P)/\rho}$, both the coefficient of q^2 and the constant term in eqn. (10) become positive and hence both q_1^2, q_2^2 become negative. Therefore q_1 and q_2 will be real if we assume c so small that it is less than both $\sqrt{(c_{11} + P)/\rho}$ and $\sqrt{(c_{44} - \frac{1}{2} P)/\rho}$.

Now u and w can be written as

$$u = \int_0^\infty \left(A_1\, e^{-kq_1z} + A_2\, e^{-kq_2z}\right) \cos k\,(x-ct)\,dk \qquad \ldots(11)$$

and

$$w = \int_0^\infty \left(A_1\, \alpha\, e^{-kq_1z} + A_2\,\beta\, e^{-kq_2z}\right) \sin k\,(x-ct)\,dk \qquad \ldots(12)$$

where

$$\alpha = \frac{(c_{44} + \frac{1}{2} P)\, q_1^2 + \rho c^2 - (c_{11} + P)}{(c_{13} + c_{44} + \frac{1}{2} P) q_1}, \quad \beta = \frac{(c_{44} + \frac{1}{2} P) q_2^2 + \rho\, c^2 - (c_{11} + P)}{(c_{13} + c_{44} + \frac{1}{2} P) q_2}. \qquad \ldots(13)$$

Writing $\delta(x-ct) = \frac{1}{\pi}\int_0^\infty \cos k(x-ct)\,dk$ and applying the boundary conditions (6) on (11) and (12) we get,

$$\left.\begin{aligned} &A_1(M\alpha - q_1) + A_2(M\beta - q_2) + \frac{F}{\pi k(c_{44}+\frac{1}{2}P)} = 0. \\ &A_1(N + \alpha q_1 c_{33}) + A_2(N + \beta q_2 c_{33}) = 0 \end{aligned}\right\} \quad \text{...(14)}$$

where

$$M = 1 - \frac{S_{33}}{c_{44}+\frac{1}{2}P}, \quad N = c_{13} + S_{33}.$$

Solving for A_1 and A_2 we get from (14)

$$\left.\begin{aligned} A_1 &= D(N + \beta q_2 c_{33})/k \\ A_2 &= -D(N + \alpha q_1 c_{33})/k \end{aligned}\right\} \quad \text{...(15)}$$

where

$$D = \frac{F}{\pi(c_{44}+\frac{1}{2}P)[(M\beta - q_2)(N+\alpha q_1 c_{33}) - (M\alpha - q_1)(N+\beta q_2 c_{33})]}. \quad \text{...(16)}$$

Substituting the values of A_1 and A_2 from (15) into (11) and (12) we have the displacement components

$$u = \int_0^\infty D\left[\frac{N + c_{33}\beta q_2}{k} e^{-kq_1 z} - \frac{N + c_{33}\alpha q_1}{k} e^{-kq_2 z}\right] \times \cos k(x-ct)\,dk \quad \text{...(17)}$$

and

$$w = \int_0^\infty D\left[\frac{N + c_{33}\beta q_2}{k}\alpha e^{-kq_1 z} - \frac{N + c_{33}\alpha q_1}{k}\beta e^{-kq_2 z}\right] \times \sin k(x-ct)\,dk. \quad \text{...(18)}$$

It is evident that q_1, q_2, α and β do not depend on k. Hence substituting the value of u and w from (17) and (18) respectively in the expressions for incremental stresses (2), and performing integration, we obtain

$$s_{11} = -D\left[\frac{(N + c_{33}\beta q_2)\{(c_{13}+P)\alpha q_1 + c_{11} + P\}}{q_1^2 z^2 + (x-ct)^2} - \frac{(N + c_{33}\alpha q_1)\{(c_{13}+P)\beta q_2 + c_{11} + P\}}{q_2^2 z^2 + (x-ct)^2}\right](x-ct) \quad \text{...(19)}$$

$$s_{33} = -D\left[\frac{(N + c_{33}\beta q_2)(c_{33}\alpha q_1 + c_{13})}{q_1^2 z^2 + (x-ct)^2} - \frac{(N + c_{33}\alpha q_1)(c_{33}\beta q_2 + c_{13})}{q_1^2 z^2 + (x-ct)^2}\right](x-ct) \quad \text{...(20)}$$

and

$$s_{31} = s_{13} = D\, c_{44} \Big[(N + c_{33}\,\beta\, q_2)\,(\alpha - q_1)\, \frac{q_1 z}{q_1^2 z^2 + (x - ct)^2} + (N + c_{33}\,\alpha\, q_1)\,(q_2 - \beta)\, \frac{q_2 z}{q_2^2 z^2 + (x - ct)^2} \Big]. \quad \ldots(21)$$

It has been seen from the expressions (19), (20) and (21) that the whole incremental stress system is moving uniformly with a velocity c in the x-direction.

The expression for shearing incremental stress shows that at any plane parallel to the boundary, the incremental shearing stress attains the maximum value at $x = ct$, i.e. at the point directly below the point of application of the shearing force on the boundary.

The maximum value at $z = z_0$ given by

$$s_{31} = \frac{D\, c_{44}}{z_0\, q_1\, q_2} \Big[(\alpha - q_1)\,(N + c_{33}\,\beta\, q_2) + (q_2 - \beta)\,(N + c_{33}\,\alpha\, q_1) \Big]. \quad \ldots(22)$$

The expression (22) shows that the maximum value of the incremental shearing stress varies inversely as depth.

It is also clear from the expressions (19) and (20) that the incremental normal stresses are zero at a point directly below the point on the boundary where the moving shearing force acts.

ACKNOWLEDGEMENT

One of the authors (A. K. Pal) is grateful to Council of Scientific and Industrial Research for financial support.

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