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SOME REMARKS ON RELATION BETWEEN NÖRLUND AND RIESZ MEANS

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The object of this paper is to obtain relations between two absolute generalised Nörlund methods (N,p,α) and (N,q,β). Our theorems obtained here generalize many known results. While discussing these inclusion relations we find that the results of Ikuko Kayashima (1973) and G. Das (1968a) become particular cases.

1. INTRODUCTION

On inclusion relations between two absolute summability methods, the following results are known.

For two given summability methods A and B throughout we use $|A| \subset |B|$ to mean that any sequence absolutely summable A to s is also absolutely summable B to s.

Theorem A—Suppose that $q_n > 0$, $Q_n \to \infty$ as $n \to \infty$*. In order that $|\bar{N},p| \subset |\bar{N},q|$ it is necessary and sufficient that $\frac{P_n}{p_n} = O\left(\frac{Q_n}{q_n}\right)$.

This result is due to Bosanquet (1950). Initially Sunouchi (1949) had given a weaker result and, in his review of Sunouchi's payer, Bosanquet pointed out that Sunouchi's argument, in fact, establishes Theorem A.

We make the following remarks regarding the results of McFadden (1942). He, in defining $|A| \subset |B|$ does not require that A and B limits should agree whereas we do. Therefore inclusions in his Theorems 2.28 and 2.29 are to be taken in the weaker sense in which the limits need not necessarily agree. But it is easily seen that under the conditions of his Theorem 2.28 limits do, in fact, agree (Theorem C below); and limits would agree in his Theorem 2.29 if we impose the additional condition that $p_n = o(P_n)$ (Theorem B below).

Theorem B—If $\{p_n\} \in \bar{M}$ and $p_n = o(P_n)$, then $|C,1| \subset |N,p|$.

Theorem C—If $\{p_n\} \in M$, then $|N,p| \subset |C,1|$.

(See § 2 for definitions of M and $\bar{M}$).

Suitably combining Theorem A with Theorem B and C we deduce Theorems 1 and 2 of Kayashima (1973) (Theorems D and E below).

Interchanging p_n and q_n in the sufficiency part of Theorem A and then putting $p_n = 1$ and combining it with Theorem B, we obtain:

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* We observe that the additional conditions that $p_n > 0$ and $p_n \to \infty$ as $n \to \infty$ assumed in Bosanquet (1950) are not needed.

Theorem D—If $q_n>0$, $\{q_n\}$ is non-decreasing, $p_n=o(P_n)$ and $\{p_n\}\in \overline{M}$, then $|\overline{N},q| \subset |N,p|$.

Next, combining Theorem C and the sufficiency part of Theorem A (with $p_n=1$), we obtain:

Theorem E—If $q_n>0$, $\{q_n\}$ is non-increasing, $Q_n\to\infty$ as $n\to\infty$ and $\{p_n\}\in M$, then $|N,p| \subset |\overline{N},q|$.

It is to be noted that inclusions in Theorems 1 and 2 of Kayashima's paper are to be taken in the weaker sense in which limits need not necessarily agree whereas in Theorems D and E limits do agree.

The purpose of this paper is to establish some inclusion relations between absolute generalised Nörlund methods. While discussing these inclusion relations we find that the results of Kayashima (1973), Das (1968a), etc. become particular cases of some of the results of this paper.

2. Definitions and Notations

Let $p=\{p_n\}$ and $\alpha=\{\alpha_n\}$ be two sequences of constants, real or complex, such that $(p * \alpha)_n\neq 0$ for all n, where

$$(p * \alpha)_n=\sum_{\nu=0}^{n} p_{n-\nu}\,\alpha_\nu .$$

Let $\sum_{n=0}^{\infty} a_n$ be a series with $\{s_n\}$ as the sequence of partial sums.

Then $\sum_{n=0}^{\infty} a_n$ is said to be summable by the generalised Nörlund method (N,p,α) to the sum s (Borwein 1958, Das 1968a) if $t_n^{p,\alpha}\to s$ as $n\to\infty$, where

$$t_n^{p,\alpha}=\frac{1}{(p*\alpha)_n}\sum_{\nu=0}^{n} p_{n-\nu}\,\pi_\nu\, s_\nu .$$

It is said to be absolutely summable (N,p,α) or summable $|N,p,\alpha|$ if $\{t_n^{p,\alpha}\}$ is a sequence of bounded variation; that is, if

$$\sum_{n=0}^{\infty}\left| t_n^{p,\alpha}-t_{n-1}^{p,\alpha}\right|<\infty,\quad (t_{-1}^{p,\alpha}=0).$$

The method (N,p,α) reduces to the Nörlund method (N,p) where $\alpha_n=1$ for all n (Hardy 1949, p. 64), and to the Riesz method $(\overline{N},\alpha)$ when $p_n=1$ for all n (Hardy 1949, p. 57).

For a sequence $\{p_n\}$, we write

$$p(z)=\sum_{n=0}^{\infty} p_n z^n$$

whenever the series on the right converges. We define the sequences of constants $\{c_n\}$ and $\{k_n\}$ by means of the following formal identities.

$$\sum_{n=0}^{\infty} c_n z^n = \frac{1}{p(z)},\ c_{-1} = 0; \qquad \ldots(1)$$

$$\sum_{n=0}^{\infty} k_n z^n = \frac{q(z)}{p(z)},\ k_{-1}=0. \qquad \ldots(2)$$

if $\{p_n\}$ satisfies $p_n>0,\ \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1$ for $n=0,1,2,\ldots,$

we shall write $\{p_n\}\in M$; and if it satisfies

$$p_n>0,\ \frac{p_{n+1}}{p_n} \geqslant \frac{p_{n+2}}{p_{n+1}} \geqslant 1 \quad \text{for } n=0,1,2,\ldots,$$

then we shall write $\{p_n\}\in \bar{M}$. If $\{p_n\}$ and $\{q_n\}$ satisfy

$$p_n>0,\ q_n>0,\ \frac{p_{n+1}}{p_n} \leqslant \frac{q_{n+1}}{q_n} \quad \text{for } n=0,1,2,\ldots,$$

then we shall write $\{p_n\} \in M\ (q_n)$.

Throughout this paper, we write for a sequence $\{p_n\}$ and an integer h,

$\delta p_n = p_n - p_{n-1}$ $(p_{-1}=0)$; $\delta^0 p_n = p_n$;

$$\delta^h p_n = \delta(\delta^{h-1} p_n)\ ;\ p_n^{(h)} = \sum_{\nu=0}^{n} p_\nu^{(h-1)}\quad p_n^{(0)} = p_n\ ;$$

$$P_n = p_n^{(1)} = \sum_{\nu=0}^{n} p_\nu.$$

3. The Lemmas

We shall require the following lemmas for the proof of our theorems.

Lemma 1—In order that the sequence-to-sequence transformation $\sigma_n = \sum_{\rho=0}^{n} d_{n,\rho}\, s_\rho$ (where throughout $d_{n,\rho}$ is taken as meaning 0 when $\rho>n$) should be such that, whenever $\{s_n\}$ converges absolutely to some limit, $\{\sigma_n\}$ converges absolutely to the same limit, it is sufficient that

$$\text{(i)}\quad D=\sum_{\rho=j}^{n}(d_{n,\rho}-d_{n-1,\rho})\geqslant 0 \quad \text{for } j=0,1,2,\ldots n;$$

(ii) $\sum_{\rho=0}^{n} | d_{n,\rho} | \leqslant K < \infty$ for all n, where K is a positive constant.

(iii) $\sum_{\rho=0}^{n} d_{n,\rho} \to 1$ as $n \to \infty$; and

(iv) $d_{n,\rho} \to 0$ as $n \to \infty$ for fixed ρ.

This is obtained by combining results of Kayashima (1973, Lemma 1) and Hardy (1949, Theorem 2).

Lemma 2 (Das 1968b, Lemma 1)—Let h be a non-negative integer such that $\{\delta^h p_n\} \in M$. Then

(i) $c_0^{(h)} > 0$, $c_n^{(h)} \leqslant 0$ for $n = 1, 2, \ldots,$;

(ii) $c_n^{(h+1)} \geqslant 0$; and

(iii) $\sum_{n=0}^{\infty} c_n^{(h)} z^n$ is absolutely convergent for $| z | \leqslant 1$.

Lemma 3—Let h be a non-negative integer. For the sequence $\{k_n\}$ to be non-negative and non-increasing it is sufficient that (i) $\{\delta^h p_n\} \in M$,

(ii) $\{\delta^h p_n\} \in M(\delta^h q_n)$ and (iii) $0 \leqslant \delta^{h+1} q_{n+1} \leqslant \dfrac{\delta^h p_{n+1}}{\delta^h p_n} \delta^{h+1} q_n$.

Remark: Das (1968c, p.168) has remarked that $\{P_n\} \in M$ and $\{q_n\} \in M\ (P_n)$ are sufficient for the condition "$\{k_n\}$ non-increasing" of his Theorem 4. But the other condition $\{p_n\} \in M''$ of his Theorem 4 and the condition $\{P_n\} \in M$ cannot be satisfied simultaneously. Hence his remark is of no avail in regard to his Theorem 4. However, the conditions of our Lemma 3 are compatible with the hypotheses of his Theorem 4.

PROOF OF LEMMA 3: Using (1), we obtain

$$\left[\sum_{n=0}^{\infty} (\delta^h p_n) z^n\right]^{-1} = \sum_{n=0}^{\infty} c_n^{(h)} z^n \qquad \ldots(3)$$

which gives
$$\sum_{\nu=0}^{n} \delta^h p_{n-\nu} c_\nu^{(h)} = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases} \qquad \ldots(4)$$

Now by (3), $\sum_{n=0}^{\infty} k_n z^n = \left[\sum_{n=0}^{\infty} q_n z^n\right]\left[\sum_{n=0}^{\infty} p_n z^n\right]^{-1}$

(*equation continued on p. 590*)

$$= \left[\sum_{n=0}^{\infty} (\delta^h q_n) z^n\right]\left[\sum_{n=0}^{\infty} (\delta^h p_n) z^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{\infty} (\delta^h q_n) z^n\right]\left[\sum_{n=0}^{\infty} c_n^{(h)} z^n\right]$$

and thus $k_n = \sum_{\nu=0}^{n} (\delta^h q_{n-\nu}) c_\nu^{(h)}$.

Using Lemma 2 and (ii), we obtain for $n=1,2,\ldots,$

$$k_n = [\delta^h q_n]\left[c_0^{(h)} - \frac{\delta^h q_{n-1}}{\delta^h q_n} \mid c_1^{(h)} \mid - \cdots - \frac{\delta^h q_0}{\delta^h q_n} \mid c_n^{(h)} \mid \right]$$

$$\geqslant [\delta^h q_n] \left[c_0^{(h)} - \frac{\delta^h p_{n-1}}{\delta^h p_n} \mid c_1^{(h)} \mid - \cdots - \frac{\delta^h p_0}{\delta^h p_n} \mid c_n^{(h)} \mid \right]$$

$$= \frac{\delta^h q_n}{\delta^h p_n} \sum_{\nu=0}^{n} (\delta^h p_{n-\nu}) c_\nu^{(h)} = 0 \text{ (by (4))}.$$

And, for $n=0$, $k_0 = (\delta^h q_0)\, c_0^{(h)} > 0$.

Thus $k_n \geqslant 0$ for all n. Next,

$$\sum_{n=0}^{\infty} (k_n - k_{n-1}) z^n = \left[\sum_{n=0}^{\infty} (\delta^{h+1} q_n) z^n\right]\left[\sum_{n=0}^{\infty} c_n^{(h)} z^n\right]$$

and so $$k_n - k_{n-1} = \sum_{\nu=0}^{n} (\delta^{h+1} q_{n-\nu})\, c_\nu^{(h)}.$$

By similar reasoning as above, and using Lemma 2 and (iii), we obtain, for $n>0$,

$$k_n - k_{n-1} \leqslant \frac{\delta^{h+1} q_n}{\delta^{h+1} p_n} \sum_{\nu=0}^{n} (\delta^h_{p_{n-\nu}})\, c_\nu^{(h)} = 0$$

by (4). This shows that $\{k_n\}$ is non-increasing and the lemma is thus proved.

4. Main Results

Theorem 1—If (i) $p_n \geqslant 0$, $\alpha_n \geqslant 0$, $k_n \geqslant 0$, (ii) $\{k_n\}$ is non-increasing (iii) either $\{q_n\}$ or $\{\alpha_n\}$ is non-decreasing, and

$$k_n = o((q * \alpha)_n), \qquad \ldots(5)$$

then $\mid N, p, \alpha \mid \subset \mid N, q, \alpha \mid$.

PROOF: Writing $t_n^{p,\alpha} = \frac{1}{(p*\alpha)_n} \sum_{\nu=0}^{n} p_{n-\nu}\, \alpha_\nu s_\nu$

we have, by a familiar inversion formula,

$$\alpha_n s_n = \sum_{\nu=0}^{n} c_{n-\nu}(p*\alpha)_\nu\, t_\nu^{p,\alpha}$$

Since, by definition, $k_n = (q*c)_n$, we have,

$$t_n^{q,\alpha} = \frac{1}{(q*\alpha)_n} \sum_{\nu=0}^{n} q_{n-\nu}\alpha_\nu s_\nu = \frac{1}{(q*\alpha)_n} \sum_{\nu=0}^{n} q_{n-\nu} \sum_{\rho=0}^{\nu} c_{\nu-\rho}(p*\alpha)_\rho\, t_\rho^{p,\alpha}$$

$$= \frac{1}{(q*\alpha)_n} \sum_{\rho=0}^{n} k_{n-\rho}\, (p*\alpha)_\rho t_\rho^{p,\alpha} = \sum_{\rho=0}^{n} d_{n,\rho} t_\rho^{p,\alpha}$$

where $d_{n,\rho} = \frac{k_{n-\rho}(p*\alpha)_\rho}{(q*\alpha)_n} (\rho \leqslant n)$.

Since, by (2), $q_n = \sum_{\nu=0}^{n} k_{n-\nu} p_\nu$, so by (i), $q_n \geqslant 0$. Thus $d_{n,\rho} \geqslant 0$. Further, if $s_n = 1$ for all n, then $t_n^{p,\alpha} = 1$, $t_n^{q,\alpha} = 1$ for all n. Hence

$$\sum_{\rho=0}^{n} | d_{n,\rho} | = \sum_{\rho=0}^{n} d_{n,\rho} = 1.$$

Next, for fixed ρ, (5) implies that

$$k_{n-\rho} = o((q*\alpha)_{n-\rho}).$$

Also, under (i) and (iii), it can be easily shown that $\{(q*\alpha)_n\}$ is non-decreasing. Therefore $(q*\alpha)_{n-\rho} \leqslant (q*\alpha)_n$ and hence

$$k_{n-\rho} = o((q*\alpha)_n).$$

Thus, for fixed ρ,

$$d_{n,\rho} \to 0 \text{ as } n \to \infty.$$

Therefore, by Lemma 1, it is enough to show that

$$D = \sum_{\rho=j}^{n} (d_{n,\rho} - d_{n-1,\rho}) \geqslant 0 \quad \text{for } j = 0,1,2, \ldots, n.$$

From (ii) and the fact that $\{(q*\alpha)_n\}$ is non-decreasing, it follows that $\left\{ \frac{k_{n-\rho}}{(q*\alpha)_n} \right\}$ is a monotonic non-increasing sequence of $n (n \geqslant \rho)$ and thus

$$d_{n-1,\rho} - d_{n,\rho} \geqslant 0 \quad \text{for } n = \rho+1, \rho+2, \ldots$$

Hence, for $0 \leqslant j \leqslant n-1$,

$$D=1-\sum_{\rho=0}^{j-1} d_{n,\rho}-1+\sum_{\rho=0}^{j-1} d_{n-1,\rho} = \sum_{\rho=0}^{j-1}(d_{n-1,\rho}-d_{n,\rho}) \geqslant 0.$$

And, for $j=n$, $D=d_{n,n}\geqslant 0$
since $d_{n-1,n}=0$. This completes the proof of the theorem.

Corollary 1 – If $\alpha_n \geqslant 0$, $q_n \geqslant 0$, $\{q_n\}$ is non-decreasing and $\{\delta q_n\}$ is non-increasing, then $|\bar{N},\alpha| \subset |N,q,\alpha|$.

This follows from Theorem 1 on putting $p_n = 1$ for all n. For, in this case,

$$c_0 = 1,\ c_1 = -1,\ c_n = 0 \text{ for } n > 1,$$

and so $k_n = (q * c)_n = q_n - q_{n-1} = \delta q_n$.
Thus, by hypotheses, $\{k_n\}$ is non-negative and non-increasing. Next, (5) reduces to

$$\delta q_n = o((q * \alpha)_n). \qquad \ldots(6)$$

Since $\{\delta q_n\}$ is non-increasing, we have

$$(q * \alpha)_n=(\delta q * \alpha^{(1)})_n \geqslant \delta q_n (\alpha_0^{(1)}+\alpha_1^{(1)}+\ldots+\alpha_n^{(1)}).$$

Also, $\alpha_n \geqslant 0$, we have $\alpha_n^{(1)} \geqslant \alpha_0$, and therefore

$$(q * \alpha)_n \geqslant \delta q_n \alpha_0 (n+1).$$

Hence (6) holds.

From Theorem 1, we deduce the following result due to Das (1968a, Theorems 4 and 5) which also includes Theorem C.

Corollary 2—Let $\{p_n\}\in M$, $\alpha_n \geqslant 0$. If either $p_n^{(1)}\to\infty$ or $\alpha_n^{(1)}\to\infty$ as $n\to\infty$, then $|N,p,\alpha| \subset |\bar{N},\alpha|$.

This follows from Theorem 1 on putting $q_n = 1$ for all n. For, in this case, $k_n = c_n^{(1)}$, so by the case $h = 0$ of Lemma 2, $\{k_n\}$ is positive and non-increasing. Also (5) reduces to

$$c_n^{(1)} = o(\alpha_n^{(1)}). \qquad \ldots(7)$$

Since $\{p_n\}\in M$, $c_n^{(1)}\to\lambda$ $(\lambda\geqslant 0)$ as $n\to\infty$ (see Das 1968a, Lemma 2). Therefore, if $p_n^{(1)}\to\infty$ (in which case $\lambda = 0$) or $\alpha_n^{(1)}\to\infty$ as $n\to\infty$, (7) clearly holds.

Remarks : In the case in which we take (iii) as "$\{\alpha_n\}$ is non-decreasing" (5) is implied by other hypotheses. For

$$(q * \alpha)_n \geqslant \alpha_0 q_n^{(1)} = \alpha_0(p^{(1)} * k)_n \geqslant \alpha_0 k_n (p_n^{(1)}+p_{n-1}^{(1)}+\ldots+p_0^{(1)}) \geqslant \alpha_0 p_0 k_n (n+1)$$

since $p_n \geqslant 0$ implies that $p_n^{(1)} \geqslant p_0$; and (5) follows.

Under the assumption "$\{q_n\}$ is non-decreasing" in (iii), a sufficient condition for (5) is that $q_n = o((q * \alpha)_n)$.

For, since $\{k_n\}$ is non-increasing and $\{q_n\}$ is non-decreasing, we must have $k_n = O(q_n)$, and (5) follows.

The assumption (5) is 'reasonable' since it is 'necessary' for the regularity of (N,q,α).

We use Theorem 1 to deduce the following more general result.

Theorem 2—If (i) $p_n \geqslant 0$, $\alpha_n \geqslant 0$, $k_n \geqslant 0$, (ii) h is a non-negative integer such that $\delta^h k_n \geqslant 0$, $\{\delta^h k_n\}$ is non-increasing, (iii) either $\{q_n\}$ or $\{\alpha_n\}$ is non-decreasing and (iv) (5) holds, then $| N,p,\alpha | \subset | N,q,\alpha |$.

PROOF : Consider first the case in which $\{\alpha_n\}$ is non-decreasing. Note that in this case, by the remark given after Corollary 2, we do not need to assume (5). If r is a non-negative integer then, applying Theorem 1 with p_n replaced by $p_n^{(r)}$ and with k_n replaced by 1 (so that q_n is replaced by $p_n^{(r+1)}$), we get

$$| N,p^{(r)},\alpha | \subset | N,p^{(r+1)},\alpha | . \qquad \ldots(8)$$

Again applying Theorem 1 with p_n replaced by $p_n^{(h)}$ and with k_n replaced by $\delta^h k_n$, we get $| N,p^{(h)},\alpha | \subset | N,q,\alpha |$. ...(9)

From (8) and (9), we deduce the required result.

Now consider the case in which $\{q_n\}$ is non-decreasing. We may suppose that $h \geqslant 1$ (since the case $h = 0$ is given by Theorem 1). We first show that, when $h \geqslant 1$, the hypotheses imply that

$$\text{either } p_n^{(1)} \to \infty \text{ or } \alpha_n^{(1)} \to \infty \text{ as } n \to \infty. \qquad \ldots(10)$$

For suppose not. Then $p_n^{(1)}$, $\alpha_n^{(1)}$ are both bounded. Now, since, $h \geqslant 1$, (ii) implies that $\{k_n\}$ is non-decreasing. Hence

$$q_n \leqslant k_n(p_0 + p_1 + \ldots + p_n) = O(k_n).$$

Similarly, since $\{q_n\}$ is non-decreasing, $(q_* \alpha)_n \leqslant q_n(\alpha_0 + \alpha_1 + \ldots + \alpha_n) = O(q_n)$. Thus $(q_* \alpha)_n = O(k_n)$, which contradicts (5).

We again establish the conclusion by proving that (8) and (9) hold. But we can no longer take it for granted that, when we make the replacements in Theorem 1 needed to obtain (8) and (9), we will still have (5) holding. Thus, in order to prove (8), we have to verify that

$$1 = o\,((p^{(r+1)}{}_* \alpha)_n). \qquad \ldots(11)$$

If $r \geqslant 1$ or if $r = 0$ and the first alternative of (10) holds, then $p_n^{(r+1)} \to \infty$ as $n \to \infty$, and thus

$$(p^{(r+1)}{}_* \alpha)_n \geqslant \alpha_0 p_n^{(r+1)} \to \infty \text{ as } n \to \infty.$$

It remains to consider the case in which $r=0$ and the second alternative of (10).

Here we have, since $\{p_n^{(1)}\}$ is non-decreasing

$$(p^{(1)} * \alpha)_n \geqslant p_0^{(1)} (\alpha_0 + \alpha_1 + \ldots + \alpha_n) \to \infty \text{ as } n \to \infty.$$

Thus, in either case, we deduce (11).

Finally, in order to prove (9), we have to verify that

$$(\delta^h k_n) = o((q * \alpha)_n).$$

But, since $\{(q * \alpha)_n\}$ is non-decreasing, this follows easily from (5). This completes the proof of the theorem.

We remark that Theorem 2 includes, as a special case, the well known result that if $\mu > \lambda \geqslant 0$, then

$$| C, \lambda | \subset | C, \mu | .$$

Theorem 3—$\alpha_n \geqslant 0$, (5) holds, and h is a non-negative integer such that (i) $\{\delta^h p_n\} \in M$, (ii) $\{\delta^h p_n\} \in M\, (\delta^h q_n)$, (iii) $0 \leqslant \delta^{h+1} q_{n+1} \leqslant \dfrac{\delta^h p_{n+1}}{\delta^h p_n} \delta^{h+1} q_n$, then

$$| N, p, \alpha | \subset | N, q, \alpha | .$$

PROOF : When $h = 0$, trivially $p_n > 0$; and (ii) implies that $q_n > 0$ and (iii) implies that $\delta q_n \geqslant 0$. Next suppose that $h > 0$. We note that

$$\delta^m p_n = \sum_{\nu=n}^{\infty} \delta^{m+1}_{p_\nu}. \qquad \ldots(12)$$

By (i), we must have $\delta^h p_n > 0$; and therefore, from (12) we successively deduce that $\delta^m p_n > 0$ with $m = h-1, h-2, \ldots, 1, 0$. Hence $p_n > 0$. Also (ii) includes the assertion that $\delta^h q_n > 0$; therefore (as proved above for p_n) $q_n > 0$ and $\delta q_n > 0$.

Thus when h is a non-negative integer, $\delta q_n \geqslant 0$ and therefore $\{q_n\}$ is non-decreasing.

Finally, by Lemma 3, $\{k_n\}$ is non-negative and non-increasing.

Thus the conditions of Theorem 1 are satisfied and the required inclusion follows.

Theorem 4—If (i) $p_n \geqslant 0$, (ii) $\{q_n\}$ is non-negative and non-increasing, (iii) $\{\alpha_n\}$ is non-negative and non-decreasing, then $| N, p, \alpha | \subset | N, p, \alpha * q, \alpha |$.

PROOF : We show that the conditions of Theorem 1 with q_n replaced by $(p * q)_n$ are satisfied. Replace q_n by $(p * q)_n$ in (2), we find that $k_n = q_n$, so, by (ii), $\{k_n\}$ is non-negative and non-increasing.

Also (5) reduces to

$$q_n = o((p * q * \alpha)_n). \qquad \ldots(13)$$

Since, by (i) and (iii), $(p * \alpha)_n$ is non-negative and non-decreasing, therefore

$$\begin{aligned}(p * q * \alpha)_n &\geqslant (p * \alpha)_0 (q_0 + q_1 + \ldots + q_n) \\ &\geqslant q_n (p * \alpha)_0 (n+1)\end{aligned}$$

by (ii), and (13) follows.

Theorem 5—If (i) $\alpha_n \geqslant 0$, $\beta_n > 0$, (ii) $\{p_n\} \in M$, (iii) $\beta_n^{(1)} \to \infty$ as $n \to \infty$

(iv) $\frac{\alpha_n^{(1)}}{\alpha_n} = O\left(\frac{\beta_n^{(1)}}{\beta_n}\right)$, then $| N, p, \alpha | \subset | \overline{N}, \beta |$.

PROOF : We assert that the hypotheses of the theorem imply that $\alpha_n^{(1)} \to \infty$ as $n \to \infty$. Suppose the assertion is false; then, since $\{\alpha_n^{(1)}\}$ is non-decreasing, it must converge. Write, for sufficiently large n,

$$\frac{\alpha_n^{(1)}}{\alpha_{n-1}^{(1)}} = 1 + u_n \ ; \ \frac{\beta_n^{(1)}}{\beta_{n-1}^{(1)}} = 1 + v_n.$$

Thus $u_n \geqslant 0$, $v_n \geqslant 0$. Now

$$\prod^{\infty} \frac{\alpha^{n}_{(1)}}{\alpha_{n-1}^{(1)}} = \prod^{\infty} (1 + u_n)$$

converge; hence $\sum_{n=1}^{\infty} u_n$ ...(14)

converegеs. Now, by (iv), there is a constant H such that

$$\frac{\beta_n}{\beta_n^{(1)}} \leqslant H \frac{\alpha_n}{\alpha_n^{(1)}},$$

which is equivalent to $\frac{v_n}{1+v_n} \leqslant H \frac{u_n}{1+u_n}$ ...(15)

The convergence of (14) implies that $u_n \to 0$ as $n \to \infty$; hence, by (15) $v_n \to 0$ as $n \to \infty$. Thus as $n \to \infty$

$$\frac{v_n}{1+v_n} \sim v_n \ ; \ \frac{u_n}{1+u_n} \sim u_n.$$

Hence (15) shows that $v_n = O(u_n)$ and thus the convergence of (11) implies that $\sum_{n=1}^{\infty} v_n$ also converges. Hence

$$\prod^{\infty} \frac{\beta_n^{(1)}}{\beta_{n-1}^{(1)}} = \prod^{\infty} (1 + v_n).$$

But this contradicts the assumption (iii) and hence the assertion is true.

The result now follows by combining Theorem A (with p_n replaced by α_n and q_n by β_n) and Corollary 2.

Writing q_n for β_n and taking $\alpha_n = 1$ for all n in Theorem 5, we obtain Theorem E. For, in this case, (iv) is implied by the assumption that $\{q_n\}$ is non-increasing.

Putting $\beta_n = 1$ for all n in Theorem 5 and combining it with Theorem B (with p_n replaced by q_n), we obtain

Theorem 6—If (i) $\alpha_n \geqslant 0$, (ii) $\{p_n\} \in M$; (iii) $\alpha_n^{(1)} = O((n+1)\alpha_n)$, (iv) $\{q_n\} \in \bar{M}$ and (v) $q_n = o(q_n^{(1)})$, then $| N, p, \alpha | \subset | N, q |$.

By putting $p_n = 1$ for all n and then writing q_n instead of α_n and p_n instead of q_n in Theorem 6, we get Theorem D.

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