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ON A MULTILATERAL GENERATING FUNCTION FOR THE EXTENDED JACOBI POLYNOMIALS

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In the present paper the authors establish a multilateral generating function involving multiple series with arbitrary terms for a general polynomial system which was introduced recently by Srivastava and Pathan (1979). Several interesting multilateral generating functions for such special polynomials as the generalised Rice and Jacobi polynomials can be derived by specializing the main result (4) which provides a multiple-series extension of a generating function due to Srivastava and Pathan [1979, p. 27, eqn. (3.1)].

1. INTRODUCTION

Recently, Srivastava and Pathan (1979) obtained certain bilateral generating functions for the extended Jacobi polynomials involving double and multiple series with arbitrary terms. In the same paper they also gave a multi-variable generalisation of such bilateral generating functions for a general polynomial system.

In an extension of this work of Srivastava and Pathan (1979) we have made an attempt to give a generalisation in the form of a multilateral generating function involving multiple series with arbitrary terms for a general polynomial system.

It is also interesting to mention here that the above mentioned result of Srivastava and Pathan (1979) follows as a special case of our main result (4).

In the course of our study, we shall require the generalised Lauricella function defined by Srivastava and Daoust [1969, p. 454; 1972, p. 158]

$$F_V^U\left[\begin{matrix}(a_p): (b'_{q_1}) ;\ldots; (b^{(r)}_{q_r}) ; \\ \\ (\alpha_P) : (\beta'_{Q_1}) ;\ldots; (\beta^{(r)}_{Q_r}) ;\end{matrix} (x_r)\right]$$

$$= \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{M_r} \cdot X}{\prod_{j=1}^{P} (\alpha_j)_{M_r} \cdot Y} \prod_{j=1}^{r} \left(\frac{x_j^{m_j}}{m_j\,!}\right), \qquad \ldots(1)$$

where (a_p) abbreviates the sequence of p parameters $a_1 ,\ldots, a_p$ with similar interpretation for (b'_{q_1}), (β'_{Q_1}), (x_r) etc., $M_r = m_1 + \ldots + m_r$, U stands for $p : q_1 ;\ldots; q_r$ and V stands for $P : Q_1 ;\ldots; Q_r$.

Also $\quad X = \prod_{j=1}^{q_1} (b'_j)_{m_1} \ldots \prod_{j=1}^{q_r} (b_j^{(r)})_{m_r}$

and $$Y = \prod_{j=1}^{Q_1} (\beta_j')_{m1} \dots \prod_{j=1}^{Q_r} (\beta_j^{(r)})_{mr} .$$

For the convergence of the multiple series (1), we must have

$$1 + P + Q_i - p - q_i \geqslant 0, \quad i = 1, 2, \dots, r; \qquad \dots(2)$$

the equality holds when, in addition,

$$\left.\begin{aligned} & p > P \text{ and } \sum_{i=1}^{r} |x_i|^{1/(p-P)} < 1 \\ \text{or} \quad & p \leqslant P \text{ and } \max\{|x_1|, \dots, |x_r|\} < 1. \end{aligned}\right\} \qquad \dots(3)$$

2. Theorem

Let $\{A_n\}, \{B_n^{(i)}\}, \{C_n^{(i)}\}$, $i = 1, 2, \dots, r$ be sequences of arbitrary complex numbers and let $\Omega(m_{r+1}^s)$, where (m_{r+1}^s) stands for $m_{r+1}, \dots, m_s$, $s > r$ be a multiple sequence of complex numbers.

Then for every complex parameter $\lambda_j \neq 0, -1, -2, \dots$ and for every positive integer s_j, $j = 1, 2, \dots, r$.

$$\sum_{n_1, \dots, n_r = 0}^{\infty} \prod_{j=1}^{r} \left[\frac{(-x_j)^{n_j}}{n_j!(\lambda_j + n_j)_{n_j}} \sum_{k_j=0}^{[n_j/s]} (-n_j)_{s_j k_j} (\lambda_j + n_j)_{s_j k_j} C_{k_j}^{(j)} \frac{w_j^{k_j}}{k_j!} \right]$$

$$\sum_{m_1, \dots, m_s = 0}^{\infty} A_{N_r + M_s} \prod_{j=1}^{r} \left\{ \frac{B_{n_j + m_j}^{(j)}}{(\lambda_j + 2n_j + 1)_{m_j}} \right\} \Omega(m_{r+1}^s) \prod_{j=1}^{s} \left(\frac{x_j^{m_j}}{m_j!} \right)$$

$$= \sum_{m_1, \dots, m_s = 0}^{\infty} A_{\sum_{i=1}^{r} (s_j - 1) m_j + M_s} \Omega(m_{r+1}^s)$$

$$\cdot \prod_{j=1}^{r} \left\{ \frac{B_{s_j m_j}^{(j)} C_{m_j}^{(j)} (w_j x_j^{s_j})^{m_j}}{m_j!} \right\} \prod_{j=r+1}^{s} \left(\frac{x_j^{m_j}}{m_j!} \right) \qquad \dots(4)$$

provided that both sides of (4) exist.

Proof : Our derivation of (4) runs parallel to that of its special case $r = 1$ given by Srivastava and Pathan [1979, p. 27, eqn. (3.1)]. Indeed, the left-hand side of (4) is equal to

$$\sum_{\substack{n_1, \dots, n_r = 0 \\ m_1, \dots, m_s = 0}}^{\infty} \sum_{k_1=0}^{[n_1/s_1]} \cdots \sum_{k_r=0}^{[n_r/s_r]} \frac{-1^{\sum_{j=1}^{r} (n_j + s_j k_j)}}{\prod_{j=1}^{r} (n_j - s_j k_j)!} A_{N_r + M_s} \Omega(m_{r+1}^s)$$

$$\times \prod_{j=1}^{r} \left\{ B_{n_j + m_j}^{(j)} C_{k_j}^{(j)} \frac{(\lambda_j)_{n_j + s_j k_j} (\lambda_j + 1)_{2n_j}}{(\lambda_j)_{2n_j} (\lambda_j + 1)_{2n_j + m_j}} \frac{w_j^{k_j} x_j^{n_j + m_j}}{k_j! \, m_j!} \right\} \prod_{j=r+1}^{s} \left(\frac{x_j^{m_j}}{m_j!} \right)$$

(*equation continued on p. 599*)

$$= \sum_{\substack{k_1,\ldots,k_r=0 \\ m_1,\ldots,m_s=0}}^{\infty} \sum_{j=1}^{A_r} s_j k_j + M_s\, \Omega\,(m_{r+1}^s) \prod_{j=1}^{r} \left\{ \frac{B_{s_jk_j+m_j}^{(j)}\; C_{k_j}^{(j)}\; (w_j\, x_j^{s_j})^{k_j}}{(\lambda_j+2s_jk_j+1)_{m_j}\, k_j\,!} \right\}$$

$$\times \prod_{j=1}^{s} \left(\frac{x_j}{m_j\,!}\right)^{m_j} \prod_{j=1}^{r} {}_3F_2 \left(\begin{matrix} \lambda_j+2s_j k_j,\ 1+s_j k_j+\frac{\lambda_j}{2} & -m_j & ; \\ & & 1 \\ s_jk_j+\frac{\lambda_j}{2}\ ,\ \lambda_j+2s_jk_j+m_j+1 & & ; \end{matrix} \right).$$

Each of the r hypergeometric series ${}_3F_2$ involved in it is well poised. By an appeal to Dixon's summation theorem [Slater 1966, p. 243, eqn. (III)], it is easily seen that

$$ {}_3F_2 \left(\begin{matrix} \lambda+2sk,\ 1+sk+\frac{\lambda}{2}\ ,\ -m\ ; \\ \qquad\qquad\qquad\qquad 1 \\ sk+\frac{\lambda}{2}\ ,\ \lambda+2sk+m+1\ ; \end{matrix} \right) = \begin{cases} 1, & \text{if } m=0 \\ 0, & \text{if } m \text{ is a positive integer.} \end{cases} \qquad \ldots(5)$$

In view of the identity (5), the result then follows.

This completes the proof of the theorem under the assumption that the various interchanges of the order of summation are permissible by the absolute convergence of the series involved.

3. Special Cases

(i) For $r = 1$, our theorem reduces to the result proved earlier by Srivastava and Pathan [1979, p. 27, eqn. (3.1)].

(ii) If we set $s_j = 1$, $j=1,2,\ldots,r$ and

$$C_{k_j}^{(j)} = \frac{\prod_{i=1}^{l_j} (d_i^{(j)})_{k_j}}{\prod_{i=1}^{t_j} (\delta_i^{(j)})_{k_j}}\ , j = 1,2,\ldots,r \qquad \ldots(6)$$

and assign special values to arbitrary coefficients A_n, B_n', $\ldots$, $B_n^{(r)}$ and $\Omega(m_{r+1}^{(s)})$ in accordance with the definition (1), we obtain the following multilateral generating function for the extended Jacobi polynomials

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{N_r X}}{\prod_{j=1}^{r} (\lambda_j+n_j)_{n_j} \prod_{j=1}^{p} (\alpha_j)_{N_r Y}} \prod_{j=1}^{r} \left(\frac{(-x_j)^{n_j}}{n_j!}\right)$$

$$\times \prod_{j=1}^{r} {}_{l_j+2}F_{t_j} \left[\begin{matrix} -n_j,\ \lambda_j+n_j,\ (d_{l_j}^{(j)}); \\ (\delta_{t_j}^{(j)}); \end{matrix}\ w_j \right]$$

(*equation continued. on p.* 600)

$$\times F^{U_1}_{V_1}\left[\begin{matrix}(a_p)+N_r: & (b'_{q_1})+n_1;\ldots; \\ (\alpha_P)+N_r: & \lambda_1+2n_1+1,(\beta'_{Q_1})+n_1;\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}(b^{(r)}_{q_r})+n_r;(b^{(r+1)}_{q_{r+1}});\ldots;((b^{(s)}_{q_s}); \\ \lambda_r+2n_r+1,(\beta^{(r)}_{Q_r})+n_r;(\beta^{(r+1)}_{Q_{r+1}});\ldots;(\beta^{(s)}_{Q_s});\end{matrix}(x_s)\right]$$

$$= F^{U_2}_{V_2}\left[\begin{matrix}(a_p):(d'_{l_1}),(b'_{q_1});\ldots;(d^{(r)}_{l_r}),(b^{(r)}_{q_r});(b^{(r+1)}_{q_{r+1}});\ldots;(b^{(s)}_{q_s}); \\ (\alpha_P):(\delta'_{t_1}),(\beta'_{Q_1});\ldots;(\delta^{(r)}_{t_r}),(\beta^{(r)}_{Q_r});(\beta^{(r+1)}_{Q_{r+1}});\ldots;(\beta^{(s)}_{Q_s});\end{matrix}(w_rx_r),(x^s_{r+1})\right], \quad \ldots(7)$$

where U_1 stands for $p:q_1;\ldots;q_s$, V_1 stands for $P:1+Q_1;\ ;1+Q_r;\ Q_{r+1};\ldots;Q_s$ and U_2 stands for $p:l_1+q_1;\ldots l_r+q_r;\ q_{r+1};\ldots;q_s$ and V_2 stands for $P:t_1+Q_1;\ ;t_r+Q_r;\ Q_{r+1};\ldots;Q_s$ the appropriate conditions as mentioned in (2) and (3) are assumed to hold.

(iii) Next if we set $l_j=1$, $t_j=2$, $\lambda_j=\alpha_j+\beta_j+1$, $d^{(j)}_1=\xi_j$, $\delta^{(j)}_1=\alpha_j+1$, $\delta^{(j)}_2=\zeta_j$, $j=1,2,\ ,r$ and specialise the generalised Lauricella function involved in (7), it yields a multilateral generating function for generalised Rice polynomials (Khandekar 1964).

(iv) If in the special case (iii) above, we further set $\xi_j=\zeta_j$ and $w_j=\frac{1}{2}(1-y_j)$, $j=1,2,\ldots,r$, we shall obtain a generating function for Jacobi polynomials (Rainville 1960, p. 254).

(v) Next, if we set $l_j=1, t_j=2$, $\lambda_j=\alpha_j+\beta_j+1, d^{(j)}_1=\xi_j$, $\delta^{(j)}_1=\alpha_j+1, \delta^{(j)}_2=\zeta_j, j=1,2,\ldots,p$ and $l_j=0$, $t_j=1$, $\lambda_j=\alpha_j+\beta_j+1$, $\delta^{(j)}_1=\alpha_j+1, w_j=\frac{1}{2}(1-y_j)$, $j=p+1,\ldots,r$ in (7), we can easily obtain a multilateral generating function involving Rice and Jacobi polynomials.

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