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A CHARACTERIZATION OF THE APPELL SET $D_n\ (x;\ a,\ k)$

S. N. SINGH AND B. K. RAI

Department of Mathematics, Banaras Hindu University, Varanasi 221005

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A multiplication formula for the Appell set of polynomials $D_n\ (x;\ a,\ k)$ has been used to characterize these polynomials, whereby the corresponding results for Bernoulli and Euler polynomials and numbers are rendered intuitive.

1. INTRODUCTION

In an attempt to incorporate the usual Bernoulli and Euler polynomials and numbers, Karande and Thakare (1975) studied the polynomials $D_n\ (x;\ a,\ k)$ introduced by means of the generating function

$$\sum_{n=0}^{\infty} D_n\,(x;\ a,\ k)\,\frac{t^n}{n!} = \frac{2\,(t/2)^k\,e^{xt}}{e^t - a} \qquad \ldots(1.1)$$

a being a non-zero real number and k an integer.

We state the following relationship between $D_n\ (x;\ a,\ k)$ and other polynomials.

(i) Bernoulli polynomials :

$D_n\ (x;\ 1,1) = B_n\ (x)$.

(ii) Euler polynomials :

$D_n\ (x;-1,0) = E_n\ (x)$.

(iii) Eulerian polynomials :

$$D_n\ (x;\ \frac{1}{\xi},\ 0) = \frac{2\xi}{\xi - 1}\ \Phi_n\ (x,\xi).$$

It may be of interest to note that these polynomials belong to Appell set of functions for

$$D_n'\ (x;\ a,\ k) = n\ D_{n-1}\ (x;\ a,\ k) \qquad (n = 0,\ 1,\ 2,\ \ldots),$$

where the prime denotes differentiation.

In this note we intend to show that (1.1) is characterized by the multiplication formula

$$\sum_{r=0}^{m-1} \frac{1}{a^r}\ D_n\ (x + \frac{r}{m};\ a^m,\ k) = \frac{m^{k-1}}{a^{m-1}}\ D_n\ (mx;\ a,\ k). \qquad \ldots(1.2)$$

2. CHARACTERIZATION

Making use of Nielsen's technique (1923), we observe that the Appell set $D_n(x;a,k)$, defined by (1.1), is characterized by the multiplication formula (1.2). More precisely, if a polynomial $f(x; a, k)$ satisfies (1.2) for a single value of $m \neq 1$, then

$f(x; a, k) = D_n(x; a, k)$.

Now let $f(x; a, k)$ be a polynomial in x such that

$$\sum_{r=0}^{m-1} \frac{1}{a^r} f(x + \frac{r}{m}; a^m, k) = \frac{m^{k-1}}{a^{m-1}} f(mx; a, k). \qquad ...(2.1)$$

We can put

$$f(x; a, k) = \sum_{s=0}^{n} A_s(a, k) D_s(x; a, k). \qquad ..(2.2)$$

It follows from (1.2), (2.1) and (2.2) that

$$\sum_{s=0}^{n} A_s(a,k) D_s(mx; a, k) = \sum_{s=0}^{n} A_s(a^m,k) D_s(mx; a, k). \qquad ...(2.3)$$

This requires that

$$A_s(a, k) = A_s(a^m, k). \qquad ...(2.4)$$

Therefore

$$F(x; a, k) = A_n(a, k) D_n(x; a, k). \qquad ...(2.5)$$

Conversely, any polynomial of the form (2.5), where the $A_n(a, k)$ are arbitrary, satisfies (2.1). In particular, if a polynomial $f(x; a, k)$ satisfies (2.1) for a single value of $m \neq 1$ then the equation holds for all m.

3. Multiplication Formula

If we define the function $\bar{D}_n(x; a, k)$ by means of

$\bar{D}_n(x; a, k) = D_n(x; a, k) \qquad (0 \leqslant x < 1)$,

$\bar{D}_n(x + 1; a, k) = a \bar{D}_n(x; a, k)$,

then it is easily seen that (1.2) holds for $\bar{D}_n(x; a, k)$, barred function, also.

Here we obtain certain generalization of (1.2) suggested by an interesting result of Mordell (1957). In extending some results of Mikolás (1957), Mordell proves the following theorem.

Let $f_1(x), \ldots, f_r(x)$ denote functions of x of period 1 that satisfy the relations

$$\sum_{s=0}^{m-1} f_i(x + \frac{s}{m}) = C_i^{(m)} f_i(mx) \qquad (i = 1, \ldots, r) \qquad ...(3.1)$$

where $C_i^{(m)}$ is independent of x. Let $a_1, \ldots, a_r$ be positive integers that are relatively prime in pairs. Then if integrals exist and $A = a_1 a_2 \ldots a_r$,

$$\int_0^A f_1(x/a_1) f_2(x/a_2) \ldots f_r(x/a_r)\, dx$$

(*equation continued on p.* 603)

$$= A \int_0^1 f_1(Ax/a_1)\ f_2(Ax/a_2) \dots f_r(Ax/a_r)\, dx$$

$$= C_1^{(a_1)}\ C_2^{(a_2)} \dots C_r^{(a_r)} \int_0^1 f_1(x)\ f_2(x) \dots f_r(x)\, dx. \qquad \dots(3.2)$$

Theorem 1—Let $r \geqslant 1$, $n_1, \dots, n_r \geqslant 1$; $a_1, \dots, a_r$ positive integers that are relatively prime in pairs; $A = a_1 a_2 \dots a_r$. Then

$$\sum_{s=0}^{mA-1} \frac{1}{a^s} \bar{D}_{n_1}\left(x_1 + \frac{s}{ma_1}; a^{ma_1}, k\right) \bar{D}_{n_2}\left(x_2 + \frac{s}{ma_2}; a^{ma_2}, k\right) \dots$$

$$\cdot\ \bar{D}_{n_r}\left(x_r + \frac{s}{ma_r}; a^{ma_r}, k\right)$$

$$= C \sum_{s=0}^{m-1} \frac{1}{a^s} \bar{D}_{n_1}\left(a_1 x_1 + \frac{s}{m}; a^m, k\right) \bar{D}_{n_2}\left(a_2 x_2 + \frac{s}{m}; a^m, k\right) \dots$$

$$\cdot\ \bar{D}_{n_r}\left(a_r x_r + \frac{s}{m}; a^m, k\right) \qquad \dots(3.3)$$

where

$$C = \frac{a_1^{k-n_1}\ a_2^{k-n_2}\ \dots\ a_r^{k-n_r}}{a^{m(a_1-1)}\ \ a^{m(a_2-1)} \dots a^{m(a_r-1)}} \qquad \dots(3.4)$$

Proof : In the first place, for $r=1$, it follows from (1.2) for arbitrary $a^* \geqslant 1$, that

$$\sum_{s=0}^{ma^*-1} \frac{1}{a^s} \bar{D}_n\left(x + \frac{s}{ma^*}; a^{ma^*}, k\right)$$

$$= \sum_{s=0}^{m-1} \sum_{p=0}^{a^*-1} \frac{1}{a^{s+mp}} \bar{D}_n\left(x + \frac{p}{a^*} + \frac{s}{ma^*}; a^{ma^*}, k\right)$$

$$= \frac{(a^*)^{k-n}}{a^{m(a^*-1)}} \sum_{s=0}^{m-1} \frac{1}{a^s} \bar{D}_n\left(a^* x + \frac{s}{m}; a^m, k\right),$$

which agrees with (3.3).

For the general case, let S denote the left member of (3.3). Put

$$A_p = a_1 a_2 \dots a_p \qquad (1 \leqslant p \leqslant r)$$

and replace s by $mp\ A_{r-1} + s$. Then

$$S = \sum_{s=0}^{mA_{r-1}-1} \frac{1}{a^s} \bar{D}_{n_1}\left(x + \frac{s}{ma_1}; a^{ma_1}, k\right) \dots \bar{D}_{n_{r-1}}\left(x_{r-1} + \frac{s}{ma_{r-1}}; a^{ma_{r-1}}, k\right)$$

(*equation continued on p. 604*)

$$\cdot \sum_{p=0}^{a_r-1} \frac{1}{a^{mp}} \bar{D}_{n_r} \left(x_r + \frac{A_{r-1}p}{a_r} + \frac{s}{ma_r} ; a^{ma_r}, k \right)$$

$$= \sum_{s=0}^{mA_{r-1}-1} \frac{1}{a^s} \bar{D}_{n_1} \left(x_1 + \frac{s}{ma_1} ; a^{ma_1}, k \right) \cdots \bar{D}_{n_{r-1}} \left(x_{r-1} + \frac{s}{ma_{r-1}} ; a^{ma_{r-1}}, k \right)$$

$$\cdot \sum_{p=0}^{a_{r-1}} \frac{1}{a^{mp}} \bar{D}_{n_r} \left(x_r + \frac{p}{a_r} + \frac{s}{ma_r} ; a^{ma_1}, k \right)$$

$$= \frac{a_r^{k-n_r}}{a^m (a_r-1)} \sum_{s=0}^{mA_{r-1}-1} \frac{1}{a^s} \bar{D}_{n_1} \left(x_1 + \frac{s}{ma_1} ; a^{ma_1}, k \right) \cdots$$

$$\cdot \bar{D}_{n_{r-1}} \left(x_{r-1} + \frac{s}{ma_{r-1}} ; a^{ma_{r-1}}, k \right) \bar{D}_{n_r} \left(a_r x_r + \frac{s}{m} ; a^m, k \right).$$

Continuing the above process, we arrive at

$$S = \frac{a_1^{k-n_1} \cdots a_r^{k-nr}}{am(a_1-1) \cdots am(a_r-1)} \sum_{s=0}^{m-1} \frac{1}{a^s} \bar{D}_{n_1} \left(a_1 x_1 + \frac{s}{m} ; a^m, k \right)$$

$$\times \bar{D}_{n_r} \left(a_r x_r + \frac{s}{m} ; a^m, k \right)$$

which completes the proof of (3.3).

4. Generalization of Multiplication Formula

If we recall an Appell set of order q defined by Hussain and Singh (1978; see also Prabhaker and Reva 1979).

$$\sum_{n=0}^{\infty} D_n^{(q)} (x; a, k) \frac{t^n}{n!} = \frac{2^q (t/2)^{kq} e^{xt}}{(e^t - a)^q},$$

where q is an arbitrary positive integer, then we have the multiplication formula

$$\sum_{s_1, \ldots, s_q = 0}^{m-1} \frac{1}{a^{s_1 + \cdots + s_q}} D_n^{(q)} \left(x + \frac{s_1 + \ldots + s_q}{m} ; a^m, k \right)$$

$$= \frac{m^{(kq-n)}}{a^{q(m-1)}} D_n^{(q)} (mx; a, k). \qquad \ldots(4.1)$$

We now define $\bar{D}_n^{(q)} (x; a, k)$ by means of

$$\bar{D}_n^{(q)} (x; a, k) = \bar{D}_n^{(q)} (x; a, k) \qquad (0 \leqslant x < 1),$$

$$\bar{D}_n^{(q)} (x+1; a, k) = a\ D_n^{(q)} (x; a, k).$$

We remark that this defintion of $\bar{D}_n(x; a, k)$ is not identical with that of Hussain and Singh (1978).

We state the following :

Theorem 2—Let $m \geqslant 1$, $r \geqslant 1$, $q \geqslant 1$; $n_1, \ldots, n_r \geqslant 0$; $a_1, \ldots, a_r$ positive integers that are relatively prime in pairs; $A = a_1 a_2 \ldots a_r$.

Then

$$\sum_{s_i=0}^{mA-1} \frac{1}{a^{s_1+\cdots+s_q}} \bar{D}_{n_1}^{(q)}\left(x_1 + \frac{s_1 + \ldots + s_q}{ma_1}; a^{ma_1}, k\right) \cdot \bar{D}_{n_2}^{(q)}\left(x_2 + \frac{s_1 + \ldots + s_q}{ma_2}; a^{ma_2}, k\right) \ldots \bar{D}_{n_r}^{(q)}\left(x_r + \frac{s_1 + \ldots + s_q}{ma_r}; a^{ma_r}, k\right)$$

$$= C^* \sum_{s_i=0}^{m-1} \frac{1}{a^{s_1+\cdots+s_q}} \bar{D}_{n_1}^{(q)}\left(a_1 x_1 + \frac{s_1 + \cdots + s_q}{m}; a^m, k\right) \cdot \bar{D}_{n_2}^{(q)}\left(a_2 x_2 + \frac{s_1 + \cdots + s_q}{m}; a^m, k\right) \ldots \bar{D}_{n_r}^{(q)}\left(a_r x_r + \frac{s_1 + \cdots + s_q}{m}; a^m, k\right), \quad \ldots(4.2)$$

where each summation is q-fold and

$$C^* = \frac{a_1^{kq-n_1} a_2^{kq-n_2} \ldots a_r^{kq-n_r}}{a^{mq(a_1-1)} a^{mq(a_2-1)} \ldots a^{mq(a_r-1)}}.$$

PROOF : The proof is very much like that of Theorem 1 and is omitted.

We remark that formulae (3.3) and (4.2), on specializing various parameters involved therein, provide multiplication formulae for products of Bernoulli, Euler and Eulerian polynomials as well as numbers due to Carlitz (1959, 1959a, 1960, 1960a).

We note also that formulae like (3.3) and (4.2) hold for any set of functions satisfying (1.2) and (4.1).

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