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ON COMMON FIXED POINTS OF MULTIMAPPINGS IN UNIFORM SPACES

S.N. MISHRA

Department of Mathematics, Garhwal University, Post Box 31,
Srinagar (Garhwal) 246174

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Let $\{F_\alpha\}$ and $\{G_\alpha\}$ be nets of multimappings of a uniform space into it's hyper-space. By considering the convergence of $\{F_\alpha\}$ and $\{G_\alpha\}$ to multimappings F and G, a common fixed point theorem is proved.

1. INTRODUCTION AND PRELIMINARIES

Wong (1973) proved a common fixed point theorem for two mappings which were not necessarily commuting nor continuous. This result was extended to the more general setting of uniform spaces for singlevalued mappings and multimappings by Mishra (1978, 1981). The purpose of this note is to prove a common fixed point theorem for a pair of multimappings which are realized as the pointwise limit of two nets of multimappings of a uniform space into it's hyper-space.

Let $(X,\mathscr{U})$ be a uniform space. A family $\{d_\lambda\}$, λ running over an index set I, of pseudometrics on X is called an associated family for the uniformity $\mathscr{U}$ if the family $\beta=\{V(\lambda,r) : \lambda\in I,\ r>0\}$ where $V(\lambda, r)=\{(x,y)\in X\times X: d_\lambda(x,y)<r\}$ is a subbase for $\mathscr{U}$. We may assume β itself to be a base by adjoining finite intersection of members of β if necessary. The corresponding family of pseudometrics is called an augmented associated family for $\mathscr{U}$. This will be denoted by P^*. It is well-known that each uniformity on X has an augmented associated family. For details the reader is referred to Thron (1966).

Let A be a nonempty subset of X. Define $\delta^*(A)=\sup\ \{d_\lambda(x,y): x,y\in A\}$ where $\{d_\lambda:\lambda\in I\}=P^*$. Further, A is to be called P^*-bounded if $\delta^*(A)<\infty$. Let 2^X denote the collection of nonempty P^*-bounded closed subsets of X. For each $\lambda\in I$ and $A,B\in 2^X$, define

$$d_\lambda(A,B)=\inf\ \{d_\lambda(a,b): a\in A,\ b\in B\}$$

$$H_\lambda(A,B)=\max\{\sup\ d_\lambda(a,B) : a\in A,\ \sup\ d_\lambda(A,b): b\in B\}.$$

Then H_λ is a pseudometric on 2^X, called the Hausdorff pseudometric induced by d_λ.

Let $U\in\mathscr{U}$ be an arbitrary entourage. For each subset A of X, let $U[A]=\{y\in X: (x,y)\in U$ for some $x\in A\}$ (cf. Kelley [1955]). The Hausdorff uniformity $2^{\mathscr{U}}$ on 2^X is defined by the base $2^{\beta}=\{U: \tilde{U}\in\mathscr{U}\}$ where $\tilde{U}=\{(A,B): A,B\in 2^x,\ A\subseteq U[B],\ B\subseteq U\ [A]\}$. The augmented associated family P^* also induces a uniformity $\mathscr{U}^*$ on 2^X defined by the base $\beta^*=\{V^*(\lambda,r),\ \lambda\in I,\ r>0\}$ where $V^*(\lambda,r)=\{(A,B): A,B\in 2^X,\ H_\lambda(A,B)<r\}$. The

uniformities $2\mathscr{U}$ and $\mathscr{U}^*$ on 2^X are uniformly isomorphic. The space $(2^X, 2^{\mathscr{U}})$ is thus a uniform space, called the hyper-space of $(x, \mathscr{U})$.

In the sequel, N will stand for the set of natural numbers. By a multimapping we mean a mapping from $(X,\mathscr{U})$ to $(2^X, 2^{\mathscr{U}})$.

The following result is due to Mishra (1981).

Theorem 1—Let $(X, \mathscr{U})$ be a complete Hausdorff uniform space defined by $\{d_\lambda : \lambda \in I\} = P^*$ and, let F, G be multimappings satisfying

$$H_\lambda(Fx,Gy) \leqslant a_\lambda d_\lambda(x,Fx) + b_\lambda d_\lambda(y,Gy) + c_\lambda d_\lambda(x,Gy) + e_\lambda d_\lambda(y,Fx) + f_\lambda d_\lambda(x,y) \quad \ldots(1.1)$$

for all $x,y \in X$ and each $\lambda \in I$ where a_λ, b_λ, e_λ, and f_λ are nonnegative real numbers such that

$$a_\lambda + b_\lambda + c_\lambda + e_\lambda + f_\lambda < 1, \quad \ldots(1.2)$$

$$a_\lambda = b_\lambda,\ c_\lambda = e_\lambda. \quad \ldots(1.3)$$

Then F and G have a common fixed point.

2. Main Result

Theorem 2—Let $(Y, \mathscr{U})$ be a complete Hausdorff uniform space defined by $\{d_\lambda : \lambda \in I\} = P^*$ and, let $\{F_\alpha\}$, $\{G_\alpha\}$, $\alpha \in N$ be nets of multimappings which converge pointwise to multimappings F and G respectively. Suppose that the pair (F_α, G_α) satisfy conditions of Theorem 1 with same constants a_λ, b_λ, c_λ, e_λ and f_λ. Suppose x_α is a common fixed point of F_α and G_α for each $\alpha \in N$. If $\{x_\alpha\}$, $\alpha \in N$ has a subnet $\{x_\nu\}$, $\nu \in N$ converging to x, then x is a common fixed point of F and G.

Proof. Let $U \in \mathscr{U}$ be arbitrary entourage. Since β is a base for $\mathscr{U}$, we can choose a $V(\lambda,r) \in \beta$ such that $V(\lambda,r) \subseteq U$.

Let $x_\nu \dashrightarrow x$. Then,

$$d_\lambda(x,Gx) \leqslant d_\lambda(x,x_\nu) + d_\lambda(x_\nu, Gx) \leqslant d_\lambda(x,x_\nu) + H_\lambda(F_\nu x_\nu, G_\nu x) + H_\lambda(G_\nu x, Gx),$$

since $x_\nu \in F_\nu x_\nu$ for all $\nu \in N$. Also, the pair (F_ν, G_ν) satisfies conditions of Theorem 1 for all $\nu \in N$, it can be easily verified that

$$d_\lambda(x,Gx) \leqslant (1 + c_\lambda + e_\lambda + f_\lambda)\, d_\gamma(x_\nu, x) + (b_\lambda + c_\lambda)\, d_\lambda(x, Gx)$$
$$+ (1 + b_\gamma + c_\lambda)\, H_\lambda(G_\nu x, Gx).$$

Therefore, $$d_\lambda(x,Gx) \leqslant \left(\frac{1 + c_\lambda + e_\lambda + f_\lambda}{1 - b_\lambda - c_\lambda}\right) d_\lambda(x_\nu, x) + \left(\frac{1 + b_\lambda + c_\lambda}{1 - b_\lambda - c_\lambda}\right) H_\lambda(G_\nu x, Gx). \quad \ldots(2.1)$$

Since $x_\nu \longrightarrow x$, for $r > 0$, there exists a positive integer m_1 such that for all $\nu \geqslant m_1$ we have

$$d_\lambda(x_\nu, x) < \left(\frac{1 - b_\lambda - c_\lambda}{1 + c_\lambda + e_\lambda + f_\lambda}\right) \cdot \frac{r}{2}. \quad \ldots(2.2)$$

Also, $G_\nu \to G$, pointwise, for given $x \in X$ and $r > 0$, there exists a positive integer m_2 such that for all $\nu \geqslant m_2$ we have

$$H_\lambda(G_\nu x, Gx) < \left(\frac{1 - b_\lambda - c_\lambda}{1 + b_\lambda + c_\lambda}\right) \cdot \frac{r}{2}. \quad \ldots(2.3)$$

Choose $m = \max\{m_1, m_2\}$. Then for all $\nu \geqslant m$, we have from (2.1), (2.2) and (2.3) that

$$d_\lambda(x,Gx) < \tfrac{1}{2}r + \tfrac{1}{2}r = r.$$

This shows that $x \in U[Gx]$ for all $U \in \mathcal{U}$. Therefore $x \in \cap U[Gx] = \bar{G}x = Gx$. Similarly it can be proved that $x \in Fx$. Therefore x is a common fixed point of F and G.

Remark 2.1 : In Theorem 2, the existence of a common fixed point x_α for F_α and G_α, $\alpha \in N$ is always guaranteed in view of Theorem 1.

Remark 2.2: If x is a complete metric space, by choosing the mappings F_α, G_α and the constants a_λ, b_λ, c_λ, e_λ and f_λ suitably, several results for singlevalued mappings and multimappings can be obtained as special cases of Theorem 2 [cf. Wong (1973, Theorem 5), Bose and Mukherjee (1980, Theorem 4.2)].

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