Indian J. pure appl. Math., 13(5), 609–613, May 1982

ON A FRACTIONAL DIFFERENTIAL OPERATOR

O. P. GARG AND K. N. ARORA

Department of Mathematics, M.M.H. College, Ghaziabad 201001

(*Received* 17 *December* 1980; *after revision* 9 *June* 1981)

In the present paper we use a fractional differential operator $D^n_{k,\alpha,x}$ to study the H-function of two variables. We establish two multiplication formulas for the H-function of two variables. As an application, we determine some known and new results for the H-function of two variables, Kampé de Fériet's function and hypergeometric functions. Many known relations are special cases of the H-function formulas. A number of known and new results for other simpler functions follow as special cases of our results.

1. INTRODUCTION

The parameters of the H-function of two variables (Mittal and Gupta 1972), occurring in the present paper, will be displayed in the following contracted notation [due essentially to Srivastava and Joshi (1969)] which was introduced formally by Srivastava and Panda [1976a, p. 266, eqn. (1.5)]:

$$H[x,y]=H^{0,n_1:m_2,n_2;\ m_3,n_3}_{p_1,q_1:p_2,q_2;\ p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,r_j)_{1,p_2};(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3}\end{matrix}\right]$$
$$=(-1/4\pi^2)\int_{L_1}\int_{L_2}\phi(s,t)\,\theta_1(s)\,\theta_2(t)\,x^s y^t\,ds\,dt \qquad \ldots(1.1)$$

where, for convenience, we let $(a_j;\alpha_j,A_j)_{n_1+1,p_1}$ and $(c_j,r_j)_{n_2+1,p_2}$ abbreviate the parameters sequences $(a_{n_1+1};\alpha_{n_1+1},A_{n_1+1}),\ldots,(a_{p_1};\alpha_{p_1},A_{p_1})$ and $(c_{n_2+1},r_{n_2+1}),\ldots,(c_{p_2},r_{p_2})$, respectively, for integers n_i,p_i, such that $0\leqslant n_i\leqslant p_i$ $(i=1,2)$, with similar interpretations for $(b_j;\beta_j,B_j)_{m_1+1,q_1}$, $(d_j,\delta_j)_{m_2+1,q_2}$, and so on [see also Gupta and Garg (1979)]. Here

$$\phi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\quad\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)}$$

$$\theta_1(s)=\frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+r_j s)\quad\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-r_j s)\quad\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)}$$

and $\theta_2(t)$ is defined analogously in terms of the parameter sets $(e_j,E_j)_{1,p_3}$, $(f_j,F_j)_{1,q_3}$.

The conditions on the parameters of the H-function of two variables, its asymptotic expansions, some of its properties, nature of contours L_1 and L_2 in (1.1), etc., can be found in Mittal and Gupta (1972).
Also

$$H^{0,\, n_1 :\, \ldots\, ;\, \ldots}_{p_1,\, q_1 :\, \ldots\, ;\, \ldots}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\, \begin{matrix} (a_j;\, \alpha_j,\, A_j)_{1,p_1} : \ldots ; \ldots \\ (b_j;\, \beta_j,\, B_j)_{1,q_1} : \ldots ; \ldots \end{matrix}\right]$$

will indicate that the parameters shown as ... are the same as those of $H[x,y]$ in (1.1), and similarly for other such notations.

2. Preliminaries

We use the fractional derivative operators defined in the following manner [see, for example, Oldham and Spanier (1974)]:

$$D_x^{\alpha}(x^{\mu-1}) = \frac{d^{\alpha}}{dx^{\alpha}}\, x^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}\, x^{\mu-\alpha-1}, \quad \alpha \neq \mu \qquad \ldots(2.1)$$

$$D_{K,\alpha,x}(x^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu+K}, \quad \alpha \neq \mu+1 \qquad \ldots(2.2)$$

where α and K are not necessarily integers.

$$D^{n}_{K,\alpha,x}(x^{\mu}) = \prod_{r=0}^{n-1}\left[\frac{\Gamma(\mu+1+rK)}{\Gamma(\mu+1+rK-\alpha)}\right] x^{\mu+nK}, \qquad \ldots(2.3)$$

which, for $K=\alpha$, becomes

$$D^{n}_{\alpha,\alpha,x}(x^{\mu}) = \frac{\Gamma[\mu+(n-1)\alpha+1]}{\Gamma(\mu+1-\alpha)}\, x^{\mu+n\alpha}. \qquad \ldots(2.4)$$

3. Fractional Derivatives of H-Functions

We obtain here a few interesting results with the help of fractional derivative operators indicated in the preceding section.

$$D^{n}_{K,\alpha,x}\left[x^{\mu}\, H^{0,\, n_1 : \ldots ; \ldots}_{p_1,\, q_1 ; \ldots ; \ldots}\left[\begin{matrix}\sigma_1 x^{\lambda} \\ \sigma_2 x^{\delta}\end{matrix}\,\middle|\,\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1} : \ldots ; \ldots \\ (b_j;\beta_j,B_j)_{1,q_1} : \ \ ; \ldots\end{matrix}\right]\right]$$

$$= x^{\mu+nK}\, H^{0,n_1+n : \ldots ; \ldots}_{p_1+n,\, q_1+n ; \ldots ; \ldots}\left[\begin{matrix}\sigma_1 x^{\lambda} \\ \sigma_2 x^{\delta}\end{matrix}\,\middle|\,\begin{matrix}(-\mu-rK;\lambda,\delta)_{r=0,n-1},(a_j;\alpha_j,A_j)_{1,p_1} : \ldots ; \ldots \\ (b_j;\beta_j,B_j)_{1,q_1},(\alpha-\mu-rK;\lambda,\delta)_{r=0,n-1} : \ldots ; \ldots\end{matrix}\right] \qquad \ldots(3.1)$$

To prove (3.1) we merely use the definitions (1.1) and (2.3) in the left-hand side of (3.1).

Particular Cases of (3.1)

By making suitable changes in the H-functions occurring in (3.1), with the help of Goyal (1975), we get, after a little simplification, the following result for the generalized Kampé de Fériet function [Appell and Kampé de Fériet 1926; see also Srivastava and Panda 1976b, p. 423, eqn. (26)]:

$$D^{n}_{K,\alpha,n}\left[x^{\mu} F^{p_1;p_2;p_3}_{q_1;q_2;q_3}\left[\begin{matrix}(a_j)_{1,p_1} : (c_j)_{1,p_2} ; (e_j)_{1,p_3} \\ (b_j)_{1,q_1} : (d_j)_{1,q_2} ; (f_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}\sigma_1 x \\ \sigma_2 x\end{matrix}\right]\right]$$

$$= x^{\mu+nK}\prod_{v=0}^{n-1}\frac{\Gamma(1+\mu+vK)}{\Gamma(1-\alpha+\mu+vK)}\, F^{p_1+n:p_2;p_3}_{q_1+n:q_2;q_3}\left[\begin{matrix}(1+\mu+vK)_{v=0,n-1},(a_j)_{1,p_1} : (c_j)_{1,p_2} ; (e_j)_{1,p_3} \\ (1-\alpha+\mu+vK)_{v=0,n-1},(b_j)_{1,q_1} : (d_j)_{1,q_2} ; (f_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}\sigma_1 x \\ \sigma_2 x\end{matrix}\right]. \qquad \ldots(3.2)$$

Also, if in (3.2), we put $p_1=p_2=p_3=q_1=1$, $q_2=q_3=0$, we get, after a little simplification, the following result:

$$D^n_{K,\alpha,x}[x^{\mu}F_1[a,c,e;\ b;\sigma_1 x,\ \sigma_2 x]$$

$$= x^{\mu+nK}\prod_{\nu=0}^{n-1}\frac{\Gamma(1+\mu+\nu K)}{\Gamma(1-\alpha+\mu+\nu K} F^{n+1:1;1}_{n+1:0;0}\left[\begin{matrix}(1+\mu+\nu K)_{\nu=0,n-1},a:c;e\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1},b:\ldots;\ldots\end{matrix}\middle|\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]$$

where F_1 is Appell's function (Appell and Kampé de Fériet 1926). Again, if in (3.3), we put $\sigma_1=\sigma_2=1$ and use Appell and Kampé de Fériet (1926,p.155), we get

$$D^n_{K,\alpha,x}[x^{\mu}{}_2F_1[a,c+e;\ b;x]]$$

$$=x^{\mu+nK}\prod_{\nu=0}^{n-1}\frac{\Gamma(1+\mu+\nu K)}{\Gamma(1-\alpha+\mu+\nu K)}\ {}_{2+n}F_{1+n}\left[\begin{matrix}(1+\mu+\nu K)_{\nu=0,n-1},a,(c+e)\\(1-\alpha+\mu+\nu K)_{\nu=0,\ n-1},\ b\end{matrix};\ x\right] \quad \ldots(3.4)$$

By making suitable changes in the H-functions occurring in (3.1) to reduce it to hypergeometric functions with the help of Goyal (1975), we get, after a little simplification, a known formula given by Misra (1975).

4. Multiplication Formulas

$$\prod_{\nu=0}^{n-1}(\mu+\nu K+a_1-1)\,H^{0,n_1+n:\ldots;\ldots}_{p_1+n,q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(1-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (a_1;\lambda,\delta),\ (a_j;\alpha_j,A_j)_{2,p_1}:\ldots;\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (b_j;\beta_j,B_j)_{1,q_1}:\ldots;\ldots\end{matrix}\right]$$

$$=H^{0,n_1+n:\ldots;\ldots}_{p_1+n,q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (a_1;\lambda,\delta),\ (a_j;\alpha_j,A_j)_{2,p_1}:\ldots\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (b_j;\beta_j,B_j)_{1,q_1}:\ldots;\ldots\end{matrix}\right]$$

$$-H^{0,n_1+n:\ldots;\ldots}_{p_1+n,q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(1-\mu-\nu K;\lambda,\delta)_{\nu=0,n-1},(a_1-1;\lambda,\delta),\ (a_j;\alpha_j,A_j)_{2,p_1}\ldots;\ldots\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (b_j;\beta_j,B_j)_{1,q_1}:\ldots;\ldots\end{matrix}\right] \quad \ldots(4.1)$$

$$\prod_{\nu=0}^{n}(\mu+\nu K+b_1)\,H^{0,n_1+n:\ldots;\ldots}_{p_1+n,\ q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(1-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},\ (a_j;\alpha_j,A_j)_{1,p_1}:\ldots;\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,\ n-1},(b_1;\lambda,\delta),\ (b_j;\beta_j,B_j)_{2,q_1}:\ldots;\ldots\end{matrix}\right]$$

$$=H^{0,n_1+n:\ldots;\ldots}_{p_1+n,\ q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(1-\mu-\nu K;\lambda,\delta)_{\nu=0,n-1},(a_j;\alpha_j,A_j)_{1,p_1}:\ldots;\ldots\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,n-1},\ (b_1+1;\lambda,\delta),\ (b_j;\beta_j,B_j)_{2,q_1}\end{matrix}\right]$$

$$+H^{0,n_1+n:\ldots;\ldots}_{p_1+n,q_1+n:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\lambda}\\ \sigma_2 x^{\delta}\end{matrix}\middle|\begin{matrix}(1-\mu-\nu K;\lambda,\delta)_{\nu=0,n-1},(a_j;\alpha_j,A_j)_{1,p_1}:\ ;\ldots\\(\alpha-\mu-\nu K;\lambda,\delta)_{\nu=0,n-1}(b_j;\lambda,\delta),(b_1;\beta_j,B_j)_{2,q_1}:\ldots;\ldots\end{matrix}\right] \quad \ldots(4.2)$$

to prove (4.1) and (4.2) we use the definitions (1.1), (2.3), (3.1) and an interesting result of gamma function $\Gamma(z+1)=z\Gamma(z)$.

5. Particular Cases of (4.1)

The following results can be deduced from (4.1) in a manner analogous to the derivation of (3.2), (3.3) and (3.4) from (3.1).

$$\prod_{\nu=0}^{n-1}(\mu+\nu K-a_1)\Gamma(\mu+\nu K)\, F^{p_1+n:p_2;p_3}_{q_1+n:q_2;q_3}\left[\begin{matrix}(\mu+\nu K)_{\nu=0,n-1},(a_j)_{1,p_1}:(c_j)_{1,p_2};(e_j)_{1,p_3}\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1},(b_j)_{1,q_1}:(d_j)_{1,q_2};(f_j)_{1,q_3}\end{matrix}\middle|\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]$$

$$=\prod_{\nu=0}^{n-1}\Gamma(1+\mu+\nu K)\, F^{p_1+n:p_2;p_3}_{q_1+n:q_2;q_3}\left[\begin{matrix}(1+\mu+\nu K)_{\nu=0,n-1},(a_j)_{1,p_1}:(c_j)_{1,p_2}(e_j)_{1,p_3}\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1},(b_j)_{1,q_1}:(d_j)_{1,q_2}:(f_j)_{1,q_3}\end{matrix}\middle|\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]$$

$$-\prod_{\nu=0}^{n-1}\Gamma(\mu+\nu K)a_1\, F^{p_1+n:p_2;p_3}_{q_1+n;q_2;q_3}\left[\begin{matrix}(\mu+\nu K)_{\nu=0,n-1},(a_1+1),(a_j)_{2,p_1}:(c_j)_{1,p_2};(e_j)_{1,p_3}\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1},(b_j)_{1,q_1};(d_j)_{1,q_2};(f_j)_{1,q_3}\end{matrix}\middle|\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right] \quad \ldots(5.1)$$

$$\prod_{\nu=0}^{n-1}(\mu+\nu K-a)\Gamma(\mu+\nu K)\,{}_{1+n}F_n\left[\begin{matrix}(\mu+\nu K)_{\nu=0,\,n-1},\ a\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1}\end{matrix};(\sigma_1+\sigma_2)x\right]$$

$$=\prod_{\nu=0}^{n-1}\Gamma(1+\mu+\nu K)\,{}_{1+n}F_n\left[\begin{matrix}(1+\mu\nu K)_{\nu=0,n-1},\ a\\(1-\alpha+\mu\nu K)_{\nu=0,n-1}\end{matrix};(\sigma_1+\sigma_2)x\right]$$

$$-\prod_{\nu=0}^{n-1}\Gamma(\mu+\nu K)a\,{}_{1+n}F_n\left[\begin{matrix}(\mu+\nu K)_{\nu=0,\,n-1},\ a+1\\(1-\alpha+\mu+\nu K)_{\nu=0,n-1}\end{matrix};(\sigma_1+\sigma_2)x\right] \quad \ldots(5.2)$$

provided $|(\sigma_1+\sigma_2)x|<1$.

Other particular cases for (4.1) can also be given on account of the most general nature of the H-function of two variables. Similarly, we can obtain particular cases for (4.2).

6. Applications

If in (4.1), we put $n=1$, $\mu=\alpha-a_1$, $\lambda=\alpha_1$ and $\delta=A_1$, we get, after a little simplification, the following formula

$$H^{0,n_1+1:\ldots;\ldots}_{p_1+1,\,q_1+1:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\alpha_1}\\ \sigma_2 x^{A_1}\end{matrix}\middle|\begin{matrix}(1-\alpha+a_1;\alpha_1,A_1),(a_j-1;\alpha_1,A_1),(a_j;\alpha_j,A_j)_{2,p_1}:\ldots:\ldots\\(a_1;\alpha_1,A_1),(b_j;\beta_j,B_j)_{1,q_1}:\ldots;\ldots\end{matrix}\right]$$

$$=H^{0,n_1:\ldots;\ldots}_{p_1,q_1:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\alpha_1}\\ \sigma_2 x^{A}\end{matrix}\middle|\begin{matrix}(q_1-\alpha;\alpha_1,A_1),(a_j,\alpha_j,A_j)_{2,p_1}:\ ;\ldots\\(b_j,\beta_j,B_j)_{1,q_1}:\ldots\ldots;\ldots\ldots\end{matrix}\right]$$

$$+(1-\alpha)H^{0,n_1:\ldots;\ldots}_{p_1,q_1:\ldots;\ldots}\left[\begin{matrix}\sigma_1 x^{\alpha_1}\\ \sigma_2 x^{A_1}\end{matrix}\middle|\begin{matrix}(1-\alpha+a_1;\alpha_1,A_1),(a_j;\alpha_j,A_j)_{2,p_1}:\ldots;\ldots\\(b_j;\beta_j,B_j)_{1,q_1}:\ldots;\ldots\end{matrix}\right] \quad \ldots(6.1)$$

which is believed to be new.

From (6.1) we can easily deduce:

$$a_1 F^{p_1+1:p_2;p_3}_{q_1+1:q_2;q_3}\left[\begin{matrix}(\alpha-1+a_1),(1+a_1),(a_j)_{2,p_1}:(c_j)_{1,p_2};(e_j)_{1,p_3}\\(a_1),(b_j)_{1,q_1}:(d_j)_{1,q_2};(f_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]$$

$$=(\alpha-1+a_1)\, F^{p_1:p_2;p_3}_{q_1:q_2;q_3}\left[\begin{matrix}(a_1+\alpha),(a_j)_{2,p_1}:(c_j)_{1,p_2};(e_q)_{1,p_3}\\(b_j)_{1,q_1}:(d_j)_{1,q_2};(f_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]$$

$$+\,(1-\alpha) F^{p_1:p_1;p_3}_{q_1:q_2:q_3}\left[\begin{matrix}(\alpha-1+a_1),(a_j)_{2,p_1}:(c_j)_{1,p_2};(e_j)_{1,p_3}\\(b_j)_{1,q_1}:(d_j)_{1,q_2};(f_j)_{1,q_3}\end{matrix}\,\middle|\,\begin{matrix}\sigma_1 x\\ \sigma_2 x\end{matrix}\right]. \quad \ldots(6.2)$$

Also, if in (5.3), we put $n=1$, $\sigma_1=\sigma_2=\frac{1}{2}$, we get, after a little simplification, the following known result given in Erdélyi *et al.* (1953):

$$(b-a)\,{}_2F_1\left({a,b \atop c};\, x\right)=b\;{}_2F_1\left({a,b+1 \atop c};\, x\right)-a\;{}_2F_1\left({a+1,b \atop c};\, x\right) \quad |x|<1. \quad \ldots(6.3)$$

ACKNOWLEDGEMENT

The authors wish to express their sincere thanks to Dr K.C. Gupta, M. R. Engineering College, Jaipur, for the encouragement given in the preparation of the present paper. Thanks are also due to worthy referee for his valuable suggestions in the improvement of the present paper. The first author (O.P.G.) is also thankful to the University Grants Commission, New Delhi, for providing financial assistance.

REFERENCES

Appell, P., and Kampé de Fériet. J. (1926). Fonctions Hypergéométriques et Hypersphériques; Polynômes d'Hermite. Gauthier-Villars. Paris.

Erdélyi, A., *et al.* (1953). Higher Transcendental Functions. Vol. 1, McGraw-Hill Book Co., Inc., New York.

Fox, C. (1961). The *G* and *H*-functions as symmetrical Fourier kernels. *Trans. Am. Math. Soc.*, **98**, 395-429.

Goyal, S.P. (1975). The *H*-function of two variables. *Kyungpook Math. J.*, **15**, 117–31.

Gupta, K.C., and Garg, O.P. (1979). On certain finite series involving the *H*-function of two variables. *Rev. Tecn.*, 2, 55–65.

Gupta, K.C , and Jain, U.C. (1966). The *H*-function, II. *Proc. Natn. Acad. Sci.* India, A36, 594–609.

Misra, A.P. (1975). On a fractional differential operator. *Ganita*, **26**, 1–18.

Mittal, P.K., and Gupta, K.C. (1972). An integral involving generalized function of two variables. *Proc. Indian Acad. Sci.*, A **75**, 117–23.

Oldham, B., and Spanier (1974). The Fractional Calculus. Academic Press, New York.

Srivastava, H.M., and Joshi, C.M. (1969). Integration of certain products associated with a generalized Meijer function. *Proc. Camb. phil. Soc.*, **65**, 471–77.

Srivastava, H. M., and Panda, R. (1976a). Some bilateral generating functions for a class of generalized hypergeometric polynomials. *J. Reine. angew. Math.*, **283/284**, 265–74.

————(1976b). An integral representation for the product of two Jacobi polynomials. *J. Lond. Math. Soc.* (2), **12**, 419–25.