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AN EXISTENCE THEOREM FOR THE NONLINEAR COMPLEMENTARITY PROBLEM

J. PARIDA AND K. L. ROY

Department of Mathematics, Regional Engineering College Rourkela 769008

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This paper is concerned with the question of the existence of a solution to $z \geqslant 0$, $F(z) \geqslant 0$ and $\langle z, F(z) \rangle = 0$ where F is a nonlinear continuous function from R^n into itself. Existence results have been obtained under varying conditions on the map F. We present existence theorems under two new conditions on F. These conditions can be seen to be more general than the previously studied ones.

1. INTRODUCTION

We shall be dealing with the nonlinear complementarity problem which is defined as follows. Let $F : R^n \to R^n$ be a continuous map, where R^n is the n-dimensional Euclidean space. Find a $z \in R^n$ such that

$$z \geqslant 0, \; F(z) \geqslant 0, \; \langle z, F(z) \rangle = 0. \qquad \ldots(1)$$

The complementarity problem has received wide attention in the mathematical programming literature since it has applications to many fields. Specifically, there are known applications of complementarity theory to linear and nonlinear programming, mathematical economics, game theory and mechanics.

One of the central problems in nonlinear complementarity theory is to characterize those F such that (1) has a solution. These characterization have been mainly based on growth conditions on F. There are a number of results, under a variety of conditions on F, which establish the existence of a solution to (1). These include the works of Cottle (1966), Karamardian (1972), More' (1974) and Megiddo and Kojima (1977). In this paper, we show, using directly a result on variational inequality, the existence of a solution to (1) under two new conditions on F. Several of the known existence theorems can be derived from the theorems presented here.

2. NOTATIONS AND PRELIMINARIES

R denotes the set of real numbers, and R^n the n-dimensional Euclidean space with the usual inner product $\langle x, y \rangle$ of $x, y \in R^n$ and norm $\| x \|$ of $x \in R^n$.

R_+ and R^n_+ denote the set of nonnegative numbers and the nonnegative orthant of R^n, respectively. By d and e, we denote any vector with all components positive and any vector with all components unity, respectively; their size follows from the context.

Let $F : R^n_+ \to R^n$, and let $G(x) = F(x) - F(0)$ for every $x \in R^n_+$. $G(x)$ is said to be positively homogeneous of some degree $\beta > 0$ if, for every $x \in R^n_+$,

$$G(\lambda x) = \lambda^{\beta} G(x) \text{ for all } \lambda \in R_+.$$

G is said to be a regular function if the system

$$0 \neq x \geqslant 0,\ G(x) + te \geqslant 0,$$
$$\langle x,\ G(x) + te \rangle = 0,\ t \geqslant 0,$$

is inconsistent. The class of regular functions is denoted by $\overline{R}$. G is a $\overline{R}_0$-function if the system

$$0 \neq x \geqslant 0,\ G(x) \geqslant 0, \langle x,\ G(x) \rangle = 0,$$

is inconsistent.

A map $F : R^n_+ \to R^n$ is a uniform P-function if there exists a $c > 0$ such that for all $x, y \in R^n_+$

$$\max_k \left\{ (x_k - y_k) \left[F_k(x) - F_k(y) \right] \right\} \geqslant c \left\| x - y \right\|^2.$$

A map $F : R^n_+ \to R^n$ is said to be strongly copositive if there exists a $c > 0$ such that for all $x \in R^n_+$

$$\langle x, F(x) - F(0) \rangle \geqslant c \left\| x \right\|^2.$$

We shall make use of the following result on variational inequalities (see Parida and Sahoo 1980).

Lemma 1 — Let C be a nonempty, compact and convex subset of R^n, and let $F : R^n \to R^n$ be continuous on C. Then there exists an $x^{\circ} \in C$ such that

$$\langle x - x^{\circ},\ F(x^{\circ}) \rangle \geqslant 0 \text{ for all } x \in C.$$

3. Main Results

Lemma 2—Let $F : R^n_+ \to R^n$ be continuous, and let $G(x) = F(x) - F(0)$ for every $x \in R^n_+$. If (1) has no solution, then there exist a sequence $\{\alpha_i\}$ of positive real numbers and a convergent sequence $\{u^i\} \subset R^n_+$ such that

(a) $\lim u^i = u,\ 0 \neq u \geqslant 0,$

(b) $u^i_k F_k(\alpha_i u^i) \leqslant 0$ for all i and $1 \leqslant k \leqslant n$,

(c) $\langle u^i, F(\alpha_i u^i)\rangle < 0$ for all i, and

(d) $F_k(\alpha_i u^i) - d_k \langle u^i, F(\alpha_i u^i)\rangle = 0$ if $u^i_k > 0$,

$F_k(\alpha_i u^i) - d_k \langle u^i, F(\alpha_i u^i)\rangle \geqslant 0$ if $u^i_k = 0$ for all i.

PROOF : For real $\alpha > 0$, consider the set

$$C_\alpha = \left\{ x : x \geqslant 0, \langle x, d \rangle \leqslant \alpha \right\}.$$

The set C_α for $0 < \alpha < \infty$ are nonempty, compact and convex, and F is continuous on each such set C_α. Hence, it follows from Lemma 1 that there exists $x^\alpha \in C_\alpha$ satisfying

$$\langle x - x^\alpha, F(x^\alpha) \rangle \geqslant 0 \quad \text{for all } x \in C_\alpha.$$

This implies that x^α solves the linear programming problem

$$\min \langle x, F(x^\alpha) \rangle \text{ over } x \geqslant 0, \langle x, d \rangle \leqslant \alpha,$$

and applying the duality theory of linear programming, we get a ξ^α such that

$$F(x^\alpha) + \xi^\alpha d \geqslant 0,\ x^\alpha \geqslant 0 \qquad \text{...(2)}$$

$$\langle x^\alpha, F(x^\alpha) + \xi^\alpha d \rangle = 0, \qquad \text{...(3)}$$

$$\left(\alpha - \langle x^\alpha, d \rangle\right) \xi^\alpha = 0,\ \langle x^\alpha, d \rangle \leqslant \alpha,\ \xi^\alpha \geqslant 0. \qquad \text{...(4)}$$

It is clear from (2) and (3) that if there is an α such that $\xi^\alpha = 0$, then x^α is a solution to (1) Therefore, we conclude that if (1) has no solution, then $\xi^\alpha > 0$ for all $0 < \alpha < \infty$. Now by (4), $\langle x^\alpha, d \rangle = \alpha$ for all these α. Let $u^\alpha = x^\alpha/\alpha$ and then $u^\alpha \geqslant 0$ and $\langle u^\alpha, d \rangle = 1$. Since the set of the points $u^\alpha, 0 < \alpha < \infty$, lies in the compact set $C = \left\{ x : x \geqslant 0, \langle x, d \rangle = 1 \right\}$, there is a convergent sequence of u^α's with $\alpha \to +\infty$. Let this sequence be one with $\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots$, or briefly, $\alpha \in \{\alpha_i\}$. Let this sequence be denoted by $\{u^i\}$, and let the vector to which the sequence converges be u. Thus we have

$$u = \lim u^i,$$

where $u \geqslant 0$ and $\langle u, d \rangle = 1$. Clearly, $u \neq 0$. This completes the proof of (a). It also follows from (2) and (3) that

$$0 < \xi^\alpha = -\frac{1}{\alpha} \langle x^\alpha, F(x^\alpha) \rangle,$$

$$F_k(x^\alpha) + \xi^\alpha d_k = 0 \quad \text{if } x^\alpha_k > 0,$$

$$F_k(x^\alpha) + \xi^\alpha d_k \geqslant 0 \quad \text{if } x^\alpha_k = 0,$$

for all $\alpha \in \{\alpha_i\}$. Now substituting $x^\alpha = \alpha u^\alpha$ in the above relations, we obtain (b), (c) and (d).

Now we give the following existence theorem for the nonlinear complementarity problem, as given by (1).

Theorem 1—Let $F: R^n_+ \to R^n$ be continuous, and assume that $g: R^n_+ \to R^n$ is a continuous map such that

$$x \geqslant 0, \left\langle x, g(x) \right\rangle \leqslant 0 \Rightarrow x = 0. \qquad \ldots(5)$$

If the map $G(x) = F(x) - F(0)$ satisfies either

$$\left\langle x, G(\lambda x) \right\rangle \geqslant c(\lambda) \left\langle x, g(x) \right\rangle \qquad \ldots(6)$$

or

$$\max_k \left\{ x_k G_k(\lambda x) \right\} \geqslant c(\lambda) \left\langle x, g(x) \right\rangle \qquad \ldots(7)$$

for some mapping $c: R_+ \to R$ with $c(t) \to +\infty$ as $t \to +\infty$, then (1) has a solution.

PROOF: Assume that (1) has no solution. Then by Lemma 2, we get a sequence $\{\alpha_i\}$ of positive real numbers and a convergent sequence $\{u^i\} \subset R^n_+$ such that

$$u = \lim u^i,\ 0 \neq u \geqslant 0,$$

$$\left\langle u^i, F(\alpha_i u^i) \right\rangle < 0 \text{ for all } i,$$

and

$$u^i_k F_k(\alpha_i u^i) \leqslant 0 \text{ for all } i \text{ and } 1 \leqslant k \leqslant n.$$

Now by (6), we obtain

$$0 > \left\langle u^i, F(\alpha_i u^i) \right\rangle \geqslant c(\alpha_i) \left\langle u^i, g(u^i) \right\rangle + \left\langle u^i, F(0) \right\rangle$$

for all i, and by (7), we obtain

$$0 \geqslant \max_k \left\{ u^i_k F_k(\alpha_i u^i) \right\} \geqslant c(\alpha_i) \left\langle u^i, g(u^i) \right\rangle - \max_k \left\{ u^i_k F_k(0) \right\}$$

for all i. Both of these imply $\left\langle u, g(u) \right\rangle \leqslant 0$ for $0 \neq u \geqslant 0$. Thus we get a contradiction to assumption (5). Hence, we conclude that (1) has a solution.

The following corollary is an easy consequence of Theorem 1.

Corollary 1—Let $F: R^n_+ \to R^n$ be continuous. Suppose that there exist a scalar $c > 0$ and a strictly copositive $n \times n$ matrix M such that

$$\left\langle x, F(x) - F(0) \right\rangle \geqslant c\,(x^T M x) \text{ for all } x \in R^n_+.$$

Then there is a solution to (1).

Remark 1: By taking $g(x) = x$, it can be shown that functions which are either strongly copositive or uniform P-functions satisfy the hypothesis of Theorem 1. In these special cases, the existence results are due ta Karamardian (1972), and More' (1974).

Theorem 2 — Let $F: R^n_+ \to R^n$ be continuous, and let $G(x) = F(x) - F(0)$ for every $x \in R^n_+$. Suppose that $G(\lambda x) = c(\lambda) G(x)$ for some mapping $c: R_+ \to R$ such that $c(t) \to +\infty$ as $t \to +\infty$. If the system

$$
\begin{aligned}
&G_k(x) + td_k = 0 \quad \text{if } x_k > 0,\\
&G_k(x) + td_k \geqslant 0 \quad \text{if } x_k = 0,\\
&t = -\left\langle x, G(x) \right\rangle \geqslant 0,\ 0 \neq x \geqslant 0,
\end{aligned} \qquad \ldots(8)
$$

is inconsistent, then (1) has a solution.

PROOF : Assume that problem (1) has no solution. By Lemma 2, we then have a sequence $\{u^i\} \subset R^n_+$ converging to a vector u such that $0 \neq u \geqslant 0$. Let $I_1 = \{k: u_k > 0\}$, and let $I_2 = \{k : u_k = 0$ and $u^i_k > 0$ for all sufficiently large $i\}$. Further, let $I = I_1 \cup I_2$. Now choose p so large that for all $i \geqslant p$, $u^i_k > 0$ if $k \in I$ and $u^i_k = 0$ if $k \notin I$. By conclusions (c) aud (d) of Lemma 2 and the hypothesis of the present theorem, we obtain

$$
\begin{aligned}
&G_k(u^i) - d_k \left\langle u^i\ G(u^i) \right\rangle + \frac{1}{c(\alpha_i)} \left\{ F_k(0) - d_k \left\langle u^i, F(0) \right\rangle \right\} = 0 \text{ if } k \in I,\\
&G_k(u^i) - d_k \left\langle u^i\ G(u^i) \right\rangle + \frac{1}{c(\alpha_i)} \left\{ F_k(0) - d_k \left\langle u^i, F(0) \right\rangle \right\} \geqslant 0 \text{ if } k \notin I \text{ and}\\
&\left\langle u^i, G(u^i) \right\rangle + \frac{1}{c(\alpha_i)} \left\langle u^i, F(0) \right\rangle < 0
\end{aligned}
$$

for all $i \geqslant p$. Taking the limit of the above relations, we find that u is a solution to system (8), which contradicts the assumption of the theorem. Hence, we conclude that (1) has a solution.

Remark 2 : It is easy to check that the hypothesis of Theorem 2 is satisfied if G is regular, and positively homogeneous of some degree $\beta > 0$. Hence, Theorem 3.1 of Karamardian (1972) and Corollaries thereof are included in Theorem 2.

As a different generalization of the existence result on homogeneous mappings, we also have the following theorem for which a proof can be easily constructed.

Theorem 3 — Let $F: R^n_+ \to R^n$ be continuous, and suppose that $\left\langle x, (F(x) \right\rangle$ is bounded below on R^n_+. Let the map $G(x) = \big(F(x) - F(0)\big)$ be a R_0- function, and let G be positively homogeneous of some degrees $\beta > 0$. Then (1) has a solution.

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