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ON THE ASYMPTOTIC REGULARITY OF CERTAIN MAPPINGS IN A BANACH SPACE

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The aim of this paper is to prove a theorem on the asymptotic regularity of certain mappings in a uniformly convex Banach space.

Let X denote a uniformly convex Banach space. A mapping $T : X \to X$ is said to be asymptotically regular at $x_0 \in X$ if $\| T^n x_0 - T^{n+1} x_0 \| \to 0$ as $n \to \infty$ whenever $T^n x_0$ is defined for all n.

Recently, Kirk (1971), Kannan (1973), and Massa (1978) have proved some results on the asymptotic regularity of certain mappings in a uniformly convex Banach space. In this paper we prove the following theorem on the asymptotic regularity of a new class of mappings in a uniformly convex Banach space.

Theorem 1—Let X be a uniformly convex Banach space, K a subset of X; and T a mapping of K into itself such that

$$\| Tx - Ty \|^2 \leqslant \alpha [\| x - Tx \| . \| y - Ty \| + \| x - Ty \| . \| y - Tx \|] + \beta [\| x - Tx \| . \| y - Tx \| + \| x - Ty \| . \| y - Ty \|] \qquad \ldots(1)$$

for all $x, y \in K$ and for nonnegative real numbers α, β with $\alpha + 2\beta \leqslant 1$. Define a mapping $S : K \to K$ by

$$S = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_n T^n, \; I = T^0,$$

where $\alpha_i \geqslant 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{n} \alpha_i = 1$. If T has at least one fixed point then the mapping S is asymptotically regular.

PROOF : Consider the sequence $\{ \| x_n - u \| \}$, where u is the unique fixed point of T in K, $x_n = S^n x_0$, $x_0 \in K$. Then we have

$$x_{n+1} - u = Sx_n - u$$

$$= \sum_{i=0}^{n} \alpha_i T^i x_n - u$$

$$= \alpha_0 (x_n - u) + (1 - \alpha_0) z_n$$

where

$$z_n = \frac{1}{(1 - \alpha_0)} \sum_{i=1}^{n} \alpha_i (T^i x_n - u).$$

From (1) we have

$$\begin{aligned} \| T^i x_n - u \|^2 &= \| T^i x_n - T^i u \|^2 \\ &\leqslant \alpha [\, \| x_n - T^i x_n \| \,.\, \| u - T^i u \| + \| x_n - T^i u \| \,.\, \| u - T^i x_n \| \,] \\ &\quad + \beta [\, \| x_n - T^i x_n \| \,.\, \| u - T^i x_n \| + \| x_n - T^i u \| \,.\, \| u - T^i u \| \,] \\ &= \alpha [\, \| x_n - u \| \;\; \| u - T^i x_n \| \,] + \beta [\, \| x_n - T^i x_n \| \,.\, \| u - T^i x_n \| \,] \end{aligned}$$

which implies

$$\| T^i x_n - T^i u \| \leqslant \left(\frac{\alpha + \beta}{1 - \beta} \right) \| x_n - u \| \leqslant \| x_n - u \| .$$

Since $\sum_{i=0}^{n} \alpha_i = 1$, it follows that

$$\| x_{n+1} - u \| = \| S x_n - u \| \leqslant \| x_n - u \| ,$$

and therefore, $\| x_n - u \| \to d_0$ for some $d_0 \geqslant 0$. If $d_0 = 0$, then $x_n \to u$ as $n \to \infty$ and so in this case

$$\| x_n - x_{n+1} \| = \| S^n x_0 - S^{n+1} x_0 \| \to 0, \text{ as } n \to \infty,$$

i.e. S is asymptotically regular at x_0. Now suppose that $d_0 > 0$. Since $\| x_n - u \| \to d_0$, $\| S x_n - u \| \leqslant \| x_n - u \|$ for each $n \geqslant 1$ and $\| S x_n - u \| = \| x_{n+1} - u \| \to d_0$ as $n \to \infty$; it follows from the uniform convexity of X that

$$\| (x_n - u) - (S x_n - u) \| \to 0, \text{ as } n \to \infty,$$

i.e.

$$\| x_n - S x_n \| = \| S^n x_0 - S^{n+1} x_0 \| \to 0, \text{ as } n \to \infty.$$

This completes the proof.

We note that the conclusion of Theorem 1 remains valid if we replace the condition (1) by

$$\begin{aligned} \| Tx - Ty \| \leqslant \alpha (\| x - y \| + \| x - Tx \| + \| y - Ty \|) \\ + (1 - 3\alpha) \max \{ \| x - y \| \; \tfrac{1}{3} (\| x - y \| + \| x \\ - Tx \| + \| y - Ty \|) \; \tfrac{1}{3} (\| x - y \| + \| x - Ty \| + \| y - Tx \|) \} \quad \ldots(2) \end{aligned}$$

for all $x, y \in K$ and $\alpha \in (0, \tfrac{1}{3}]$.

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