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ON ALMOST PARA CONTACT METRIC MANIFOLDS: NIJENHUIS TENSOR

R. D. S. KUSHWAHA AND D. K. YADAV

Department of Geophysics, Banaras Hindu University, Varanasi 221005

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In this paper, we have expressed the Nijenhuis tensor in various forms in almost para contact metric manifolds and para contact metric manifolds.

1. INTRODUCTION

Let us consider an $(n+1)$-dimensional real differentiable manifold M with fundamental tensor field F of type (1,1), fundamental vector field T and 1-form A such that for any vector-field X we have (Sato 1976)

$$\overline{\overline{X}} = X - A(X)T \text{ for an arbitrary vector field } X \qquad \text{...(1.1a)}$$

where

$$\bar{X} \;\underline{\underline{\text{def}}}\; F(X) \qquad \text{...(1.1b)}$$

$$A(T) = 1 \qquad \text{...(1.2)}$$

$$\bar{T} = 0 \qquad \text{...(1.3)}$$

$$A(\bar{X}) = 0. \qquad \text{...(1.4)}$$

Then M is called an almost para contact manifold. An almost para contact manifold M is said to be an almost para contact metric manifold if a Riemannian metric tensor g satisfies (Sato 1976)

$$g(\bar{X},\bar{Y}) = g(X,Y) - A(X)A(Y) \qquad \text{...(1.2a)}$$

$$g(T,X) = A(X). \qquad \text{...(1.2b)}$$

If we put

$$'F(X,Y) \;\underline{\underline{\text{def}}}\; g(\bar{X},Y). \qquad \text{...(1.3)}$$

Then we have

$$'F(X,Y) = 'F(Y,X) \qquad \text{...(1.4a)}$$

$$'F(\bar{X},\bar{Y}) = 'F(X,Y) \qquad \text{...(1.4b)}$$

$$'F(X,\bar{Y}) = 'F(\bar{X},Y). \qquad \text{...(1.4c)}$$

The Nijenhuis tensor of F in an almost para contact metric manifold is a vector valued bilinear scalar function N given by

$$N(X,Y) \;\underline{\underline{\text{def}}}\; [\bar{X},\bar{Y}] + \overline{\overline{[X,Y]}} - \overline{[X,\bar{Y}]} - \overline{[\bar{X},Y]}. \qquad \text{...(1.5)}$$

Using (1.1a), we have in an almost para contact metric manifold

$$N(X,Y) = [\bar{X},\bar{Y}] + [X,Y] - \overline{[\bar{X},Y]} - \overline{[X,\bar{Y}]} - A([X,Y])T. \qquad \text{...(1.6)}$$

Almost para contact metric manifold is called para contact metric manifold if Nijenhuis tensor vanishes.

Theorem 1.1—Let us put

$$H(X,Y) = [\bar{X},\bar{Y}] + [X,Y]. \quad \text{...(1.7)}$$

Then

$$H(X,Y) - \overline{H(\bar{X},Y)} = N(X,Y) + A(X)\,\overline{[T,\bar{Y}]} + A([X,Y])T. \quad \text{...(1.8)}$$

PROOF : Barring X in (1.15) and using (1.1a), we obtain

$$H(\bar{X},Y) = \overline{[X,\bar{Y}]} - A(X)\,\overline{[T,\bar{Y}]} + \overline{[\bar{X},Y]}.$$

Barring the whole equation and using (1.1a), we obtain

$$\overline{H(\bar{X},Y)} = \overline{[X,\bar{Y}]} - A(X)\overline{[T,\bar{Y}]} + \overline{[\bar{X},Y]}.$$

Subtracting this equation from (1.7) and (1.6). we get

$$H(X,Y) - \overline{H(\bar{X},Y)} = N(X,Y) + A(X)\,\overline{[T,\bar{Y}]} + A([X,Y])T$$

which is (1.8)

Corollary 1.1—The equation (1.8) equivalent to

$$N(T,Y) = H(T,Y) - \overline{[T,\bar{Y}]} - A([T,Y])T. \quad \text{...(1.9)}$$

PROOF : Putting T for X in (1.8) and using (1.3), we get (1.9).

Theorem 1.2—Let us put

$$Q(X,Y) = [\bar{X},\bar{Y}] - \overline{[\bar{X},Y]}. \quad \text{...(1.10)}$$

Then

$$Q(X,Y) - \overline{Q(\bar{X},Y)} = N(X,Y) + A(X)\,\overline{[T,\bar{Y}]} - A(X)\,[T,Y] + A(X)\,A([T,Y])T. \quad \text{...(1.11)}$$

PROOF : Barring X in (1.10) and using (1.1a). we obtain

$$Q(\bar{X},Y) = [X,\bar{Y}] - A(X)\,\overline{[T,Y]} - \overline{[X,Y]} + A(X)\,\overline{[T,Y]}.$$

Barring the whole equation and using (1.1a). we obtain

$$\overline{Q(\bar{X},Y)} = \overline{[X,\bar{Y}]} - A(X)\,\overline{[T,\bar{Y}]} - [X,Y] + A([X,Y])T - A(X)\,[T,Y] - A(X)\,A([T,Y])T.$$

Subtracting this equation from (1.10) and using (1.6), we get

$$Q(X,Y) - \overline{Q(\bar{X},Y)} = N(X,Y) + A(X)\,\overline{[T,\bar{Y}]} - A(X)\,[T,Y] + A(X)\,A([T,Y])T$$

which is (1.11).

Corallary 1.2—In almost para contact metric manifold we have

$$N(T,Y) = Q(T,Y) - \overline{[T,\bar{Y}]} + [T,Y] - A([T,Y])T. \quad \text{...(1.12)}$$

PROOF: Putting T for X in (1.11) and using (1.3), we get (1.12).

Theorem 1.3—Let us put

$$P(X,Y) = [\bar{X},Y] - \overline{[X,\bar{Y}]}. \quad \text{...(1.13)}$$

Then

$$P(X,Y) - \overline{P(X,\bar{Y})} = N(X,Y) + A(Y)\,\overline{[\bar{X},T]} - A(Y)\,([X,T]) + A(Y)\,A([X,T])T. \quad \text{...(1.14)}$$

PROOF: Barring Y in (1.13) and using (1.1a), we obtain

$$P(X,\bar{Y}) = [\bar{X},Y] - A(Y)\,([\bar{X},T]) - \overline{[X,Y]} + A(Y)\,\overline{[X,T]}.$$

Barring the whole equation and using (1.1a), we get

$$\overline{P(X,\bar{Y})} = \overline{[\bar{X},Y]} - A(Y)\overline{[\bar{X},T]} - [X,Y] + A([X,Y])T + A(Y)[X,T] - A(Y)A([X,T])T.$$

Subtracting this resulting equation from (1.13) and (1.6) we obtain

$$P(X,Y)-\overline{P(X,\bar{Y})}=N(X,Y)+A(Y)\,\overline{[\bar{X},T]}-A(Y)\,[X,T]+A(Y)\,A([X,T])T$$

which is (1.14).

Corollary 1.3—In para contact metric manifold, we have

$$N(X,T)=P(X,T)-\overline{[\bar{X},Y]}+[X,T]-A[X,T]T \qquad ...(1.15)$$

PROOF: Putting T for Y in (1.14) and using (1.3), we get (1.13).

Theorem 1.4—In almost para contact metric manifold, we have

$$Q(T,Y)-H(T,Y)=[T,Y]. \qquad ...(1.16)$$

PROOF: From (1.12) and (1.9) by equating the right-hand sides, we at once get (1.16).

Theorem 1.5—In almost para contact metric manifold, we have

$$P(X,Y)-\overline{(P(X,\bar{Y})}=A(Y)\overline{[\bar{X},T]}-A(Y)[X,T]+A(Y)\,A([X,T])T+Q(X,Y)-\overline{Q(\bar{X},Y)}$$
$$-A(X)\overline{[T,\bar{Y}]}+A(X)[T,Y]-A(X)\,A([T,Y])T. \qquad ...(1.17)$$

PROOF: From (1.14) and (1.1a) we atonce get (1.17).

Theorem 1.6—In almost para contact metric manifold, we obtain

$$P(X,Y)-\overline{P(X,\bar{Y})}=A(Y)\overline{[\bar{X},T]}-A(Y)[X,T]+A(Y)A([X,T])T+H(X,Y)$$
$$-\overline{H(\bar{X},Y)}-A(X)(\overline{[T,\bar{Y}]})-A([X,Y])T. \qquad ...(1.18)$$

PROOF: From (1.14) and (1.8), we at once get (1.18)

Theorem 1.7—We have, in para contact metric manifold

$$TA([\bar{X},\bar{Y}])=0. \qquad ...(1.19a)$$

PROOF: Barring X and N in (1.6) and using (1.1a), we obtain

$$\overline{N(\bar{X},Y)}=\overline{[X,\bar{Y}]}-A(X)\overline{[T,\bar{Y}]}+\overline{[\bar{X},\bar{Y}]}-[\bar{X},\bar{Y}]+A([\bar{X},\bar{Y}])T$$
$$+A([X,Y])T-[X,Y]+A(X)[T,Y]-A(X)A([T,Y])T. \qquad ...(1.20)$$

Using (1.6) and

$$N(T,Y)=[T,Y]-\overline{[T,\bar{Y}]}-A([T,Y])T$$

in (1.20), we obtain

$$\overline{N(\bar{X},Y)}=-N(X,Y)+A(X)N(T,Y)+A([\bar{X},\bar{Y}])T. \qquad ...(1.21)$$

For para contact metric manifold it reduces to

$$A([\bar{X},\bar{Y}])T=0$$

which is (1.19a).

Theorem 1.7a—We have, in para contact metric manifold

$$A(X)\{[T,Y]-[T,\bar{Y}]\}-A([\bar{X},Y])T=A(Y)\{\overline{[X,T]}-[\bar{X},T]\}\;\;A([X,\bar{Y}])T. \qquad ...(1.22)$$

PROOF: Barring X and Y in (1.6) respectively and subtracting from each other and using (1.1), we obtain

$$N(\bar{X},Y)+A(X)\,\{\overline{[T,\bar{Y}]}\}-[T,\bar{Y}]\}-A([\bar{X},Y])T=N(X,\bar{Y})+A(Y)\{\overline{[X,T]}-[\bar{X},T]\}$$
$$-A([X,\bar{Y}])T. \qquad ...(1.23)$$

For para contact metric manifold $N(X,Y)=0$ so, we atonce obtain (1.22).

Corollary 1.4—We have in a para contact metric manifold

$$A([\bar{X},\bar{Y}])=0 \qquad ...(1.24)$$

$$A([X,\overline{Y}]) = A(X)A([T,\overline{Y}]). \qquad \ldots(1.25)$$

PROOF: (1.24) follows from (1.21). Barring X in (1.24) and using (1.4) and (1.1a) we obtain (1.25).

Corollary 1.5—We also have in a para contact metric manifold

$$[\overline{X},T]-\overline{[X,T]}=A[\overline{X},T]T \qquad \ldots(1.26)$$

$$A(X)A[(T,\overline{Y})] + A([\overline{X},\overline{Y}])=A(Y)A([\overline{X},T])-A([X,Y]). \qquad \ldots(1.27)$$

PROOF: Putting T for Y in (1.22) and using (1.3), we obtain (1.26). (1.27) follows from (1.25).

Theorem 1.8—We have in almost para contact metric manifold

$$\overline{N(X,\overline{Y})}=-N(X,Y)+A([\overline{X},\overline{Y}])T+A(Y)\,N(X,T) \qquad \ldots(1.28a)$$

$$\overline{N(X,\overline{Y})}-\overline{N(\overline{X},Y)}=A(Y)\,N(X,T)-A(X)\,N(T,Y). \qquad \ldots(1.28b)$$

PROOF: Barring Y and N in (1.14) and using (1.1), we obtain

$$\overline{N(X,\overline{Y})}=\overline{[\overline{X},Y]}-A(Y)[\overline{X},\overline{T}]+\overline{[X,\overline{\overline{Y}}]}-[X,Y]+A([X,Y])T+A(Y)([X,T])$$
$$-A(Y)A([X,T])T-[\overline{X},\overline{Y}]+A([\overline{X},\overline{Y}])T. \qquad \ldots(1.29)$$

Using (1.6) and

$$N(X,T)=[X,T]-\overline{[\overline{X},T]}-A([X,T])T \qquad \ldots(1.30)$$

in (1.29), we obtain (1.28a).

By subtracting (1.21) from (1.28a), we obtain (1.28b).

Corollary 1.6—We have, in an almost para contact metric manifold

$$A(N(X,Y))=A(Y)\,[A(N(X,T))+A(\overline{X},\overline{Y}]) \qquad \ldots(1.31)$$

$$A(X)A(N(T,Y))=A(Y)A(N(X,T)). \qquad \ldots(1.32)$$

PROOF: Using (1.4) in (1.28a), we obtain (1.31). Using (1.4) in (1.28b), we obtain (1.32).

Corollary 1.7—We also have in an almost para contact metric manifold

$$A(N(T,Y))=0.$$

PROOF : Using (1.4) in (1.21), we obtain (1.33).

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