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CHARACTERISTIC NUMBERS FOR ORIENTED SINGULAR G-BORDISM

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The aim of the paper is to develop the characteristic numbers for oriented singular principal G-manifolds and to prove their invariance with regard to oriented singular G-bordism.

1. INTRODUCTION

Thom (1954) gave the notion of Stiefel-Whitney numbers and proved their invariance with respect to unoriented bordism. The oriented case involves the Pontrjagin numbers also and was tackled by Wall (1960). Lee and Wasserman (1972) developed these notions for G-manifolds. In our previous paper (Khare and Sharma 1976) we have developed characteristic numbers for unoriented singular principal G-manifolds, G being a finite group and proved their invariance with regard to unoriented singular G-bordism. The object of the present paper is to give the notion of characteristic numbers for oriented singular principal G-manifolds and to prove their invariance with regard to oriented singular G-bordism.

2. PRELIMINARIES AND CHARACTERISTIC NUMBERS

Throughout the paper, by a manifold we will mean a compact oriented differentiable manifold. Let G be a group and (X,A) be a topological G-pair. Consider the set $N_n(X, A; G)$ consisting of all triples $(M^n, f; G)$, where M^n is an n-dimensional manifold with free G-action and $f: M^n \to X$ is an equivariant map with $f(\partial M^n) \subseteq A$, ∂M^n being the boundary of M^n. Consider a relation $\sim$ in $N_n(X, A; G)$ given by $(M_1^n, f_1; G) \sim (M_2^n, f_2; G)$ if and only if $\exists$ an $(n+1)$-dimensional principal G-manifold W^{n+1} with an equivariant map $F: W^{n+1} \to X$ such that

(a) $\exists$ an orientation-preserving equivariant diffeomorphism of $M_1^n \cup (-M_2^n)$ in $\partial\delta W^{n+1}$, $-M_2^n$ denoting the manifold M_2^n with the reversed orientation and $\cup$ denoting the disjoint union

(b) $F \big| (M_1^n \cup (-M_2^n)) = f_1 \cup f_2$, where $f_1 \cup f_2;\ (M_2^n \cup (-M_2^n)) \to X$ is defined as

$$(f_1 \cup f_2)(x) = \begin{cases} f_1(x), & \text{if } x \in M_1^n \\ f_2(x), & \text{if } x \in M_2^n \end{cases}$$

(c) $F\left(W^{n+1} - (M_1^n \cup (-M_2^n))\right) \subseteq A$.

It is easy to see that $\sim$ is an equivalence relation in $N_n(X, A; G)$. We denote the set of equivalence classes thus obtained by $\Omega_n(X, A; G)$ and the equivalence class containing the triple $(M^n, f; G)$ by $[M^n, f; G]$. $\Omega_n(X, A; G)$ is an abelian group with respect to the following binary operation

$$\left[M_1^n, f_1; G\right] + \left[M_2^n, f_2; G\right] = \left[M_1^n \cup M_2^n, f_1 \cup f_2; G\right].$$

The group $\Omega_n(X, A; G)$ is called the n-dimensional oriented singular G-bordism group of the pair (X, A). If $A = \phi$, we denote $\Omega_n(X, A; G)$ by $\Omega_n(X; G)$.

Consider the universal G-bundle $\xi : EG \to BG$ and the action of G on $X \times EG$ given by $g(x, y) = (gx, gy)$, $\forall (x, y) \in X \times EG$ and $g \in G$. Let $[M^n, f; G]$ be an element of $\Omega_n(X, A; G)$. Consider the following commutativ ediagram

$$\begin{array}{ccc} M^n & \xrightarrow{\alpha} & EG \\ \Big\downarrow \eta & & \Big\downarrow \xi \\ M^n/G & \xrightarrow{\beta} & BG \end{array}$$

where M^n/G is the orbit manifold, β is the classifying map of the principal G-bundle $\eta : M^n \to M^n/G$ and α is the equivariant map covering β. Consider (f, α) $M^n \to X \times EG$ defined by $(f, \alpha)(x) = (f(x), \alpha(x))$. Then $[M^n/G, \overline{(f, \alpha)}]$ is an element of the n-dimensional oriented singular bordism group $\Omega_n((X \times EG)/G, (A \times EG)/G)$, where $\overline{(f, \alpha)} : M^n/G \to (X \times EG)/G$ is the map obtained from (f, α) on passing to quotients. This gives a map

$$\mu : \Omega_n(X, A; G) \to \Omega_n\left((X \times EG)/G, (A \times EG)/G\right)$$

defined by

$$\mu\left(\left[M^n, f; G\right]\right) = \left[M^n/G, \overline{(f, \alpha)}\right].$$

By straight forward verification, one gets the following :

Proposition 2 1— μ is an isomorphism.

Let X be a CW-complex with a finite group G acting on X such that X/G be again a finite CW-complex. Let h^* be an equivariant cohomology and h_* be the associated equivariant homology (Bredon 1967). Suppose $h^* = H^* \circ A$ and $h_* = H_* \circ A$, where A is a functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle\, ,\, \rangle : h^*(X) \otimes_{H^*(pt)} h_*(X) \to H_*(pt)$$

be the Kronecker pairing. Let us assign to each G-manifold M, a class $[M, \partial M] \in h_*(M, \partial M)$ such that

(a) $[M_1 \cup M_2, \partial M_1 \cup \partial M_2] = [M_1, \partial M_1] + [M_2, \partial M_2]$ and
(b) $\partial_* [M, \partial M] = [\partial M]$.

The class $(M, \partial M)$ belonging to $h_* (M, \partial M)$ is called a 'topological class' of the G-manifold M.

Suppose $[M^n, f; G] \in \Omega_n (X; G)$ and $x \in h^* (B (SO, G)_n)$, $B (SO, G)_n$ being the classifying space [Khare 1977, page 12] for oriented G-vector bundles of dimension n. We define $\left\langle \tau^*_{M^n} (x) \cup f^* (a^m), [M^n] \right\rangle \in H_* (pt.)$ to be the 'x-characteristic number of the map $f: M^n \to X$ associated with an element $a^m \in h^m (X)$', where $\tau_{M^n} : M^n \to B (SO, G)_n$ is the tangent map (the dimension of $\tau^*_{M^n} (x) \cup f^* (a^m)$ should be n).

3. Invariance of Characteristic Numbers

Throughout the section, we will be considering the following equivariant cohomology and homology

$$h^* (X) = H^* \Big((X \times EG)/ G; \mathbb{Z} \oplus \mathbb{Z}_2 \Big)$$

and
$$h_* (X) = H_* \Big((X \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2 \Big).$$

Theorem 3.1 — If $[M^n, f; G]$ is the zero element in $\Omega_n (X; G)$, then all the x-characteristic numbers of the map $f : M^n \to X$ associated with each $a^m \in h^m (X)$ are zero.

Proof : Since G acts freely on M^n

$$\begin{aligned} h_* (M^n) &\overset{q_*}{\approx} H_* (M^n/ G; \mathbb{Z} \oplus \mathbb{Z}_2) \\ &\approx H_* (M^n/G; \mathbb{Z}) \oplus H_* (M^n/G; \mathbb{Z}_2), \end{aligned}$$

where q_* is the isomorphism induced from the homotopy equivalence $q : M^n/G \to (M^n \times EG)/G$ given by $q[x] = [x, \alpha (x)]$, α being the map from M^n to EG covering the classifying map $\beta : M^n/G \to BG$ of the principal G-bundle $M^n \to M^n/G$. Since M^n/G is an oriented manifold, $h_*(M^n)$ has a topological class of M^n say σ_n in dimension n. Since $[M^n, f; G]$ is zero in $\Omega_n (X; G)$, $\exists$ an $(n + 1)$-dimensional principal G-manifold W^{n+1} with an equivariant map F: $W^{n+1} \to X$ such that $\partial W^{n+1} = M^n$ and $F | M^n = f$. Let $[W^{n+1}, \partial W^{n+1}]$ be a topological class of W^{n+1}. Then $\partial_*[W^{n+1}, \partial W^{n+1}] = \sigma_n$. We have the following commutative diagram :

$$\begin{array}{ccc} M^n = \partial W^{n+1} & \xrightarrow{\ \tau_{M^n}\ } & B (SO, G)_n \\ \big\downarrow i & \curvearrowright & \big\downarrow j \\ W^{n+1} & \xrightarrow{\ \tau_{W^{n+1}}\ } & B (SO, G)_{n+1} \end{array}$$

where $j : B (SO, G)_n \to B (SO, G)_{n+1}$ is the map classifying $\gamma^n_s G \oplus 1$, $\gamma^n_s (G)$:

$E(SO,G)_n \to B(SO,G)_n$ being the universal oriented G-vector bundle of dimension n. Also

$$h^*(B(SO,G)_n) = H^*\big((B(SO,G)_n \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2\big)$$
$$\approx H^*(BG \times BSO_n; \mathbb{Z} \oplus \mathbb{Z}_2)$$

by Dieck (1969). Applying Künneth formula, we have

$$h^*(B(SO,G)_n) \overset{\theta^*}{\approx} H^*(BG; \mathbb{Z} \oplus \mathbb{Z}_2) \otimes H^*(BSO_n; \mathbb{Z} \oplus \mathbb{Z}_2).$$

Similarly

$$h_*(B(SO,G)_n) \overset{\theta_*}{\approx} H_*(BG; \mathbb{Z} \oplus \mathbb{Z}_2) \otimes H_*(BSO_n; \mathbb{Z} \oplus \mathbb{Z}_2).$$

Since the map j^* is surjection, for every $x \in h^*(B(SO,G)_n) \ \exists\ y \in h^*(B(SO,G)_{n+1})$ such that $j^*(y) = x$. Therefore

$$\begin{aligned}
&\left\langle \tau^*_{M^n}(x) \cup f^*(a^m), \sigma_n \right\rangle \\
&= \left\langle \tau^*_{M^n}\, j^*(y) \cup f^*(a^m), \sigma_n \right\rangle \\
&= \left\langle (j\, \tau_{M^n})^*(y) \cup i^* F^*(a^m), \partial_*[W^{n+1}, \partial W^{n+1}] \right\rangle \\
&= \left\langle i^*\, \tau^*_{W^{n+1}}(y) \cup i^* F^*(a^m), \partial_*[W^{n+1}, \partial W^{n+1}] \right\rangle \\
&= \left\langle \tau^*_{W^{n+1}}(y) \cup F^*(a^m), i_*\partial_*[W^{n+1}, \partial W^{n+1}] \right\rangle = 0,
\end{aligned}$$

since $i_*\partial_* = 0$. This completes the proof of the theorem.

Conversely we prove the following :

Theorem 3.2 — If all the x-characteristic numbers of an equivariant map f: $M^n \to X$ associated with every $a^m \in h^m(X)$ is zero, then $[M^n, f; G]$ is zero in $\Omega_n(X,G)$ provided that all the torsion elements of $H^*((X \times EG)/G; \mathbb{Z})$ have order two.

PROOF : By Proposition 2.1, we have $\Omega_n(X;G) \approx \Omega_n((X \times EG)/G)$. Therefore it is enough to show that $[M^n/G, (f\alpha)]$ is zero. We are given that $\left\langle \tau^*_{M^n}(x) \cup f^*(a^m), [M^n] \right\rangle = 0$, $x \in h^{n-m}(B(SO,G)_n)$ and $a^m \in h^m(X)$.

Also one has the following commutative diagrams

$$\begin{array}{ccc}
h^*(B(SO,G)_n) & \xrightarrow{\ \tau^*_{M^n}\ } & h^*(M^n) \\
\downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle q^*} \\
H^*(BG; \mathbb{Z} \oplus \mathbb{Z}_2) \otimes H^*(BSO_n; \mathbb{Z}\oplus\mathbb{Z}_2) & \xrightarrow{\ \overline{\tau^*_{M^n}}\ } & H^*(M^n/G; \mathbb{Z} \oplus \mathbb{Z}_2)
\end{array}$$

and

$$\begin{array}{ccc} h^*(x) & \xrightarrow{\quad f^* \quad} & h^*(M^n) \\ \Big\| & & \Big\downarrow q^* \\ H^*((X \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2) & \xrightarrow{\quad \bar{f}^* \quad} & H^*(M/G; \mathbb{Z} \oplus \mathbb{Z}_2) \end{array}$$

where $\overline{\tau_M^{*n}}$ and $\bar{f}^*$ are the maps obtained from the tangent map τ_M^{*n} and the map f^* on passing to quotients and q^* is the isomorphism induced from the homotopy equivalence $q : M^n/G \to (M^n \times EG)/G$ given by $q[x] = [x, \alpha(x)]$. Also $\overline{\tau^*_{M^n}} = (\beta \times k)^*$, where $k : M^n/G \to BSO_n$ is the classifying map for the n-dimensional bundle $T(M^n)/G \to M^n/G$, $T(M^n)$ being the tangent manifold on M^n.

Let $\bar{\sigma}_n$ be the fundamental class of M^n/G. Then $\sigma_n = q_*(\bar{\sigma}_n)$.

Therefore $\left\langle \tau_M^{*n}(x) \cup f^*(a^m), \sigma_n \right\rangle = 0$

$$\left\langle (q^*)^{-1} (\beta \times k)^* \theta^*(x) \cup (q^*)^{-1} \bar{f}^*(a^m), q_*(\bar{\sigma}_n) \right\rangle = 0$$

$$\left\langle (\beta \times k)^* \theta^*(x) \cup \bar{f}^*(a^m), \bar{\sigma}_n \right\rangle 0$$

$$\left\langle (\beta \times k)^*(y) \cup \bar{f}^*(a^m), \bar{\sigma}_n \right\rangle = 0$$

$\forall\, y \in H^*(BG; \mathbb{Z} \oplus \mathbb{Z}_2) \otimes H^*(BSO_n; \mathbb{Z} \oplus \mathbb{Z}_2)$ and $a^m \in H^m((X \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2)$. Also it is simple to see that $T(M^n)/G = T(M^n/G)$. Therefore the Stiefel-Whitney classes of M^n/G can be expressed in teams of $k^*(\omega_1), \ldots, k^*(\omega_n)$ where ω_i is the ith Stiefel-Whitney class of the universal bundle $\gamma_s^n : ESO_n \to BSO_n$. This shows that all the Stiefel-Whitney classes of M^n/G are in the image of $(\beta \times k)^*$. Thus

$$\left\langle W_1^{i_1} \cup \ldots \cup W_n^{i_n} \cup \bar{f}^*(a^m), \bar{\sigma}_n \right\rangle = 0,$$

$\forall\, a^m \in H^m((X \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2)$ with $i_1 + 2i_2 + \ldots + ni_n = n - m$, W_j being the jth Stiefel-Whitney class of M^n/G. Further all the Pontrjagin classes of M_n/G too can be expressed in terms of the images of the Pontrjagin classes of the universal bundle γ_s^n under k^*. Hence

$$\left\langle p_1^{i_1} \cup \ldots \cup p_r^{i_r} \cup \bar{f}^*(a^m), \bar{\sigma}_n \right\rangle = 0,$$

$\forall\, a^m \in H^m((X \times EG)/G; \mathbb{Z} \oplus \mathbb{Z}_2)$, $r \leqslant n/4$ with $4(i_1 + \ldots + ri_r) = n - m$, p_j being the jth Pontrjagin class of M^n/G. Thus we conclude that $[M^n/G, \overline{(f, \alpha)}]$ has all the Stiefel-Whitney and Pontrjagin numbers of $\overline{(f, \alpha)} : M^n/G \to (X \times EG)/G$ equal to zero. We are already given that all the torsion elements of $H^*((X \times EG)/G; Z)$ have order two. Therefore by Theorem 17.5 of Conner and Floyd (1964), we infer that $[M^n/G, (\overline{f, \alpha})]$ is zero in Ω_n $((X \times EG)/G)$ which completes the proof of the theorem.

Theorems 3.1 and 3.2 give the following :

Theorem 3.3 — If X is a finite CW-complex with a finite group G acting on X such that $(X \times EG)/G$ is again a finite CW-complex and if all the torsion elements of H^* $((X \times EG)/G;\mathbb{Z})$ have order two, then an element $[M^n, f; G]$ in Ω_n $(X; G)$ is zero if and only if all the x-characteristic numbers of the map $f: M^n \to X$ associated with every element $a^m \in h^m$ (X) are zero for every $x \in h^*$ $(B\,(SO,G)_n)$.

Taking X to be a point space with the trivial action of G, we have Theorem 5 of (Wasserman and Lee 1972).

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