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ON BEST AND BEST SIMULTANEOUS APPROXIMATION

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In this note it has been shown that if G is an approximatively compact set in a metric space (X, d) then the set of best approximations in G to an element x in X is a compact set and the set G is a retraction of X. Further a relationship between best approximation and best simultaneous approximation in metric linear spaces has been established.

1. INTRODUCTION

The notion of approximatively compact set was introduced by Efimov and Steckin (1961) to study the problem of best approximation. Singer (1970) has shown that if G is a non-void approximatively compact set in a metric space (X, d) then G is proximinal i.e. for each $x \in X$, the set

$\alpha_G(x) = \{g \in G \mid d(x, G)\} = d(x, G)\}$ is non-void.

Some properties of this set have been already proved (cf. Singer 1970). Here, we have shown (Theorem 2.1) that $\alpha_G(x)$ is compact if G is approximatively compact. We have also proved (Theorem 2.2) that an approximatively compact Cebysev set in a metric space (X, d) is a retraction of X. Finally, we have established (Theorem 3.1) a relationship between best approximation and best simultaneous approximation in metric linear spaces by showing that if M is a subspace of a metric linear space (X, d) in which orthogonality is homogeneous then every pair x_1, x_2 in $M^{\perp}$ has a best simultaneous approximation in M which is also a best approximation of $\frac{1}{2}(x_1 + x_2)$ if x_1 and x_2 are linearly dependent.

We recall here the following definitions:

Definition 1.1—A set G in a metric space (X, d) is said to be 'approximatively compact' if for each $x \in X$ and every sequence $\langle g_n \rangle$ in G with

$$\lim_{n\to\infty} d(x, g_n) = d(x, G)$$

there exists a subsequence $\langle g_{n_k} \rangle$ converging to an element of G.

Definition 1.2—A set G in a metric space (X,d) is said to be 'proximinal' if for each $x \in X$ there exists a point $g \in G$ which is nearest to x i.e.

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$\alpha_G(x) = \{g \in G : d(x,g) = d(x, G)\}$ is non-empty for each $x \in X$. If $\alpha_G(x)$ consists of exactly one point for each $x \in X$ then G is called a 'Cebysev set'.

Definition 1.3—If G is a proximinal set in a metric space (X,d) then the set-valued mapping $P_G(x)$ defined on X, which takes each point x of X to the set $\alpha_G(x)$ is called the 'nearest point map' or metric projection. Clearly $P_G(g) = g$ for each $g \in G$.

Definition 1.4—A subset G of a metric space (X,d) is said to be a 'retraction' of X (notion due to Borsuk 1931) if there exists a continuous function $r : X \to G$ such that $r(y) = y$ for all $y \in G$.

Definition 1.5—An element x of a metric linear space (X,d) is said to be 'orthogonal' to another element $y \in X$ [notion introduced in Narang (1976)] $x \perp y$ if

$$d(x,0) \leqslant d(x, \lambda y) \text{ for each scalar } \lambda.$$

If M is a subspace of a metric linear space (X,d), we define $M^{\perp}$ as

$$M^{\perp} = \{x \in X : x \perp g \text{ for all } g \in M\}.$$

Orthogonality in X is said to be 'homogeneous' if $x_1 \in M^{\perp}$ implies $\alpha x_1 \in M^{\perp}$ for every scalar α.

Definition 1.6—Let G be a subspace of a metric space (X,d). An element $g_0 \in G$ is said to be a 'best simultaneous approximation' of the pair $x_1,x_2 \in X$ if

$$\max \{d(x_1, g_0), d(x_2, g_0)\} = \inf_{g \in G} \max \{d(x_1, g), d(x_2,g)\}.$$

2. Approximatively Compact Sets and Best Approximation

The following theorem shows that the set of best approximations in G to an element $x \in X$, i.e. the set $\alpha_G(x)$, is compact if G is approximatively compact.

Theorem 2.1—If G is an approximatively compact set in a metric space (X,d) then the set $\alpha_G(x)$, is compact.

Proof: Let $\langle g_n \rangle$ be a sequence in $\alpha_G(x)$. This means that

$$d(x,g_n) = d(x,G) \text{ for all } n \text{ and so}$$

$$\lim_{n \to \infty} d(x,g_n) = d(x,G).$$

Since G is approximatively compact, $\langle g_n \rangle$ has a subsequence $\langle g_{n_k} \rangle$ converging to an element g^*. As $\alpha_G(x)$ is closed (cf. Singer 1970), $g^* \in \alpha_G(x)$.

Corollary 2.1—$\alpha_G(x)$ is compact if G is spherically compact.

This follows from the fact that every spherically compact set in a metric space is approximatively compact.

Corollary 2.2—$\alpha_G(x)$ is compact if G is a boundedly compact closed set.

This follows from the result that every boundedly compact closed set in a metric space is approximatively compact (cf. Singer 1970).

Next we show that an approximatively compact set in a metric space (X,d) is a retraction of X. For compact spaces this result was proved by Kurtowski (1936).

Theorem 2.2—An approximatively compact Cebysev set in a metric space (X,d) is a retraction of X.

PROOF : Let G be an approximatively compact Cebysev set in (X,d). Consider the nearest point map $P_G(x)$.

Since $P_G(x)$ is continuous (The nearest point map onto an approximatively compact Cebysev set in a metric space is continuous (cf. Singer 1970) and $P_G(g) = g$ for all $g \in G$, G is a retraction of X.

3. Best Approximation and Best Simultaneous Approximation

The problem of best approximation and best simultaneous approximation has been studied by many investigators (cf. Singer 1970, Ahuja and Narang 1979). We have derived a relationship between these two notions in metric linear spaces. [In normed linear spaces this result was given by Muthukumar (1980)].

Theorem 3.1—Let M be a subspace of metric linear space (X,d). Then every pair $x_1,x_2 \in M^\perp$ has a best simultaneous approximation in M which is also a best approximation of the arithmetic mean of x_1,x_2 if x_1,x_2 are linearly dependent and orthogonality in X is homogeneous.

PROOF : Let $x_1,x_2 \in M^\perp$ and let $d(x_1,0) \geqslant d(x_2,0)$
(The case $d(x_2,0) \geqslant d(x_1,0)$ is similar).
Then

$$\begin{aligned} \max\{d(x_1,0),d(x_2,0)\} &= d(x_1,0) \\ &\leqslant d(x_1,\alpha g) \text{ for all } g \in M \text{ as } x_1 \in M^\perp \\ &\leqslant \max\{d(x_1,\alpha g), d(x_2,\alpha g)\} \text{ for all } g \in M \end{aligned}$$

This implies

$$\begin{aligned} \max\{d(x_1,0),d(x_2,0)\} &\leqslant \inf_{g\in M} \max\{d(x_1,\alpha g), d(x_2,\alpha g)\} \\ &\leqslant \max\{d(x_1,0), d(x_2,0)\} \end{aligned}$$

Thus

$$\max\{d(x_1,0), d(x_2,0)\} = \inf_{g\in M} \max\{d(x_1,\alpha g), d(x_2,\alpha g)\}$$

i.e. 0 is a best simultaneous approximation to x_1 and x_2.

Now if x_1 and x_2 are linearly dependent then

$\frac{x_1+x_2}{2} = \frac{x_1+\lambda x_1}{2} = \frac{(1+\lambda)\,x_1}{2}$ for some scalar λ. Since 0 is a best approximation of every element $x_1 \in M^\perp$ (cf. Narang 1976) and as orthogonality in X is homogeneous, $x_1 \in M^\perp$ implies that $\frac{(1+\lambda)}{2}\,x_1 \in M^\perp$ and so 0 is best approximation

to $\frac{(1+\lambda)}{2}\,x_1$.

i.e. to $\frac{x_1+x_2}{2}$.

Remark 1 : The homogeneity of orthogonality is necessary in Theorem 3.1. It follows from the result (cf. Narang 1976). "Let G be a subspace of a metric linear space (X,d) $x \in X \mid \overline{G}$ and $g_0 \in G$. Then g_0 is a best approximation to x if and only if $x-g_0 \perp G$" (0 is a best approximation to $\frac{1}{2}(1+\lambda)\,x_1$ if and only if $\frac{1}{2}(1+\lambda)\,x_1 - 0 \in M^\perp$ i.e. if and only if $\frac{1}{2}(1+\lambda)\,x_1 \in M^\perp$).

Remark 2 : The first part of Theorem 3.1 can be generalised for n, $n > 2$ elements, whether the second part of the theorem also holds for linear metric spaces is not known to the authors. However, this does hold in inner product spaces. A proof similar to that given in Muthukumar (1980, Theorem 3.1), can be easily constructed.

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